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DECEMBER 1961

The AMERICAN MATHEMATICAL MONTHLY

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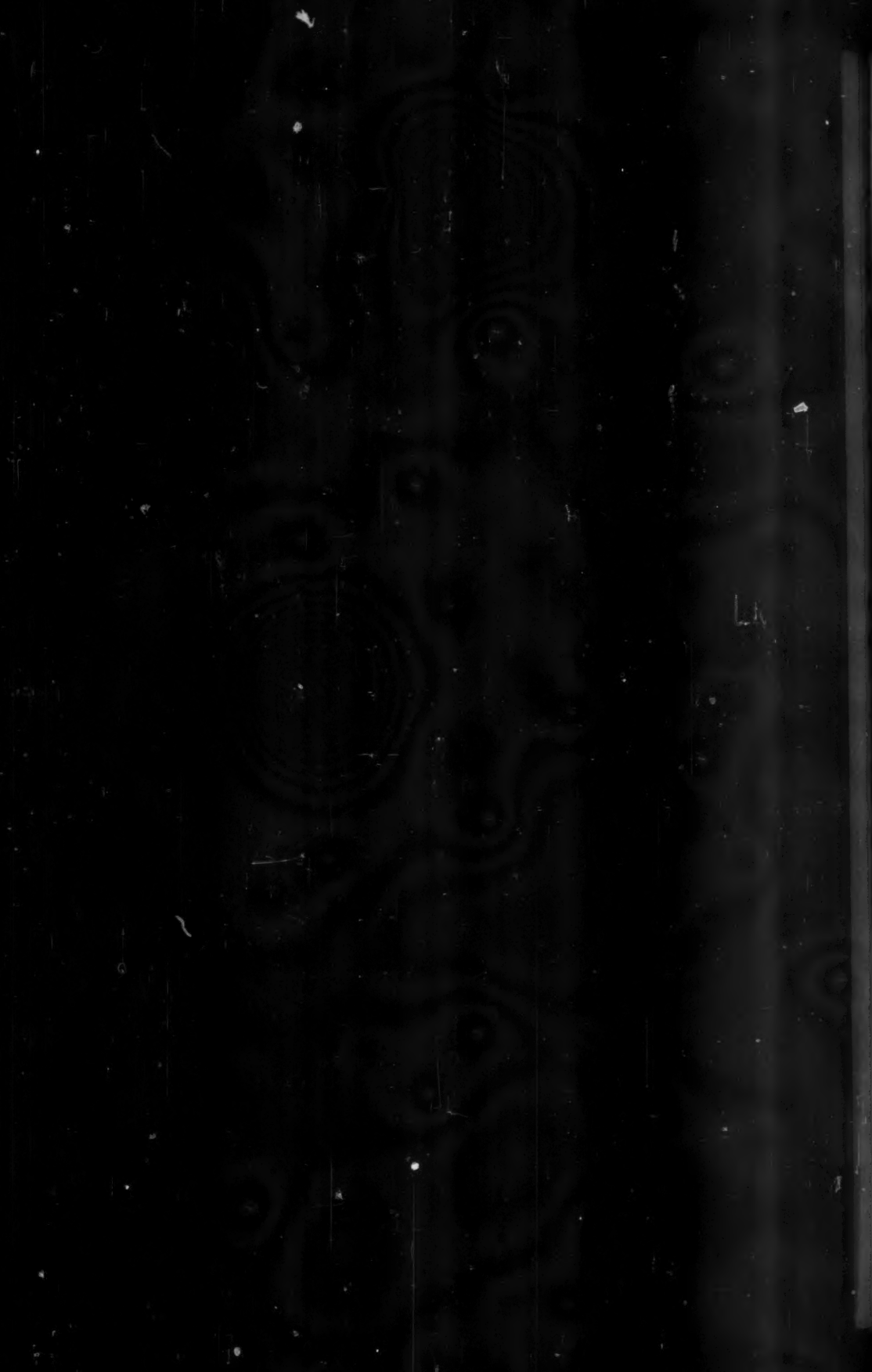

INVERTIBLE SPACES

P. H. DOYLE AND J. G. HOCKING, Michigan State University

Our characterization of the n-sphere in terms of inversion [2] led us to define and study the concept of topological invertibility. While modest in scope, invertibility is typical of many new developments and provides an easily accessible example of mathematical research. In the hope that both the results and the means of discovery of those results will be of interest, we present this rambling account of our investigation.

To show how our new subject started, we must describe the characterization of the n-sphere at least in part. Suppose that U is a nonempty open subset of the n-sphere S^n (and we mean the geometric set $\{x: |x| = 1\}$ in E^{n+1}). Regardless of the size and shape of U, we can find an $(n-1)$-sphere S^{n-1} centered at any point of U and lying inside of U. An inversion of S^n in S^{n-1} (essentially a reciprocal radii transformation) interchanges the two components of the complement $S^n - S^{n-1}$ of S^{n-1}, leaving S^{n-1} fixed. Since such an inversion is a one-to-one, continuous mapping of S^n onto itself, the following statement is true:

For each nonempty open set U in S^n there is a homeomorphism h of S^n onto itself with the property that the image $h(S^n - U)$ of the complement of U lies in U.

When we found that the n-sphere is the only n-manifold with this property (see Th. 1 of [2]), an obvious question arose. Are there any other topological spaces (not manifolds, of course) with this same kind of topological invertibility? We immediately noted that the rational points on the n-sphere are inverted (essentially) by the inversion of the n-sphere. It follows immediately that the rational points in n-space constitute a space which may be inverted topologically. This example is not compact, however, and we wanted a compact example.

Suppose one wishes to study compact subspaces of E^1 which contain no 1-cells. The first example which comes to mind is, perhaps, the Cantor set. And we leave it as a simple exercise to show that the Cantor set has the desired topological invertibility. This immediate success in uncovering examples convinced us that a property which characterizes the n-sphere among n-manifolds would be worthy of investigation. A name was chosen and the formal definition set up.

A topological space S is *invertible* if for each nonempty open subset U of S there is a homeomorphism h of S onto itself such that $h(S-U)$ lies in U. The homeomorphism h is called an *inverting homeomorphism* for the open set U.

In this terminology the characterization of the n-sphere given in [2] may be stated as "The n-sphere is the only invertible n-manifold." Indeed, in [2] we show that if an invertible space is locally Euclidean at any point, then it is a sphere.

The first property of the general invertible space we located was part of the proof of our characterization theorem.

THEOREM 1. *If the invertible space S contains a nonempty open set U whose closure $\overline{U}$ is compact, then S is compact.*

Proof. An inverting homeomorphism h for the open set U will carry the closed complement $S-U$ in U. Then, as a closed subset of the compact set $\overline{U}$, $h(S-U)$, and hence $S-U$, is compact. Therefore S is the union of the two compact sets $\overline{U}$ and $S-U$.

COROLLARY. *A locally compact invertible space is compact.*

This very easy result immediately led us to ask for a comparable result involving connectedness in place of compactness. Attempting to prove such a conjecture led quickly to the following result.

THEOREM 2. *If S is an invertible space and contains an open connected set U, then S consists of at most two components. If S is not connected, then U and $S-U$ are the components of S and they are homeomorphic.*

Proof. An inverting homeomorphism h for U carries $S-U$ into U and, since h is onto, $S-U$ must lie in $h(U)$. If S were not connected, then the connected open set would be separated unless $h(U)=S-U$. Thus if S is not connected, the components of S are U and $S-U$ and they are homeomorphic.

It was necessary to introduce a separation property in order to complete the proof of the desired analogue of Theorem 1.

THEOREM 3. *If the invertible space S is a T_1-space and contains a nondegenerate, open, connected set, then S is connected.*

Proof. If S is not connected, then by Theorem 2, U is a component of S and $S-U$ is homeomorphic to U. Since S is a T_1-space, the removal of a point p from U leaves an open set $U-p$. Then there is an inverting homeomorphism g for $U-p$ and $g(S-U)$ lies in $U-p$. But then $g(S-U)$ is a component of S properly contained in the component U. This is contradictory.

COROLLARY. *A locally connected invertible T_1-space either is connected or is the zero-sphere.*

(We are still amused by the complicated characterization of the zero-sphere implicit in this corollary.)

It was at this stage of our investigation that we formulated the following heuristic principle: *If an invertible space has a given local property, then it also has the corresponding global property.* Note that this is not a theorem.

The next two results are typical of those suggested by this principle and there are many we omit because they grow wearisome.

THEOREM 4. *If the invertible space S has a nonempty open set U which, as a subspace, is a T_i-space, $i=0, 1, 2$, then S is a T_i-space.*

Proof. Suppose that U is a T_0-space and let p and q be points of S. If p and q lie in U, the separation of p and q can be made since U is a T_0-space. If p lies in U and q in $S-U$, then the separation of p and q is already accomplished.

Finally, if both points lie in $S-U$, then an inverting homeomorphism for U carries the points into U where the desired separation can be made. If either of the stronger separation properties is assumed, the proof is just as easy.

THEOREM 5. *If the invertible T_1-space S is locally arcwise connected, then S is arcwise connected.*

Proof. By definition, given any point p of an open set U, there is an open set V contained in U and containing p such that every pair of points in V can be joined by an arc lying in U. Let x and y be any two points in S. If x and y lie in V, the desired arc is already assumed. If both x and y lie in $S-V$, an inverting homeomorphism h for V carries x and y into V, an arc joining them is chosen and then the entire arc carried back with the inverse h^{-1} of the homeomorphism h. If x is in V and y in $S-V$, we choose an inverting homeomorphism for the open set $V-x$ and play the same game.

The addition of the T_1 separation property to the hypotheses of our theorems is apparently unavoidable. That is, the property of invertibility seems to have no relation to the separation properties themselves. One of the first examples of a T_1-space that is not a T_2 (Hausdorff)-space which a student might see consists of a countably infinite set S in which a subset is defined to be open if and only if its complement is finite. It is easy to show that a one-to-one mapping of S onto itself which moves only a finite number of points is a homeomorphism. It follows that S is invertible; indeed, this is an example of a connected, compact invertible T_1-space which is not T_2.

In discussing our work here we noted how often we said "map the closed set C into the open set U." This observation led to the following result.

THEOREM 6. *A space S is invertible if and only if, for each proper closed subset C and each nonempty open set U, there is a homeomorphism h of S onto itself such that $h(C)$ lies in U.*

Proof. Certainly a space with the assumed property is invertible. Hence we assume that S is invertible. Given the sets U and C as in the hypotheses, if C lies in $S-U$, an inverting homeomorphism for U satisfies the desired conclusions. If $U-C$ is not empty, then it is open and an inverting homeomorphism for $U-C$ satisfies the theorem. Of course, if $U-C$ is empty, an inverting homeomorphism for $S-C$ followed by one for U yields a homeomorphism h such that $h(C)$ lies in U.

An immediate application of Theorem 6 to T_1-spaces (in which points are closed) showed that an invertible T_1-space S has the following property: *If x is a point of S and U is an open set of S, then there is a homeomorphism h of S onto itself such that $h(x)$ lies in U.* This property has been called *near homogeneity* by Burgess [1] but we were unaware of this name when we found the following:

THEOREM 7. *In a connected invertible T_1-space S the set of cut points either is empty or is dense in S.*

Proof. Let p be a cut point of S if such exists. Then by Theorem 6 there is homeomorphic image of p in every open set in S. But the property of being a cut point is invariant under homeomorphisms and hence every open set contains a cut point.

It will be noted that no more than the near homogeneity of S is used in the above proof and so Theorem 7 may be generalized in the obvious way. The same is true about the next theorem which was the product of an obvious comment about Theorem 7. The remark? Well, one of us said "What's so special about cut points in this kind of space?"

Let $G(S)$ denote the group of homeomorphisms of the space S onto itself. If x is a point of S, then the *orbit of* x, O_x, is the set of all images of x under elements of $G(S)$.

LEMMA 8. *The orbit of each point in an invertible space is dense in the space.*

The proof of Lemma 8 is precisely the same as that of Theorem 7, of course.

This lemma started an investigation of the orbits of points. By the very definition, an orbit O_x is homogeneous, that is, given any two points y and z in O_x, there is a homeomorphism in $G(S)$ carrying y onto z. Furthermore, if we consider such an orbit as a subspace, it is easy to see that an orbit is itself an invertible space.

LEMMA 9. *If S is invertible and x is any point in S, then the orbit O_x of x is a homogeneous invertible subspace of S.*

THEOREM 10. *Every invertible space is the union of disjoint homogeneous invertible subspaces, each dense in the space.*

Proof. Combine Lemmas 8 and 9 with the very easily proved fact that the orbits are equivalence classes.

We had only the following example of a nonhomogeneous invertible space at the time we obtained Theorem 10. Let S consist of an open interval of real numbers and a point p not in the interval. A subset of S is open if and only if its complement is a countable subset of the interval. Thus each open set contains the point p. It is easy to prove that this is an invertible T_0-space which is not T_1. The point p has for its orbit only itself (the point p *is* dense in S) and the open interval is the orbit of any other point.

Having Theorem 10, it seemed necessary to construct an example in which there were at least two nondegenerate orbits. Theorem 7 provided a clue and we looked for a continuum which had a dense set of cut points and a dense set of noncut points. One such continuum which is invertible may be constructed in the plane as follows: Start with a single closed interval of unit length. At the midpoint of this interval erect an interval of length $\frac{2}{3}$ so that the two intervals are perpendicular bisectors of each other. At the midpoint of each of the four intervals in the resulting figure, erect an interval whose length is $\frac{2}{3}$ that of the

interval being bisected. If the single closed interval is the 0th stage, then we will have 4^n intervals in the nth stage. Stages 1, 2, and 3 are pictured in Figure 1.

FIG. 1

The continuum C described above may be seen to be invertible by the following considerations: The four components of any branch point of order four in C all are homeomorphic. An explicit construction can be given in every case for this homeomorphism. Now given any open set U in C, we may choose a branch point p of order four such that two of the four components of $C-p$ lie entirely in U. Then interchanging these two with the two that do not lie in U is an inverting homeomorphism.

The example above contains three orbits, (i) the collection of noncut points of C, (ii) the collection of branch points of order four and (iii) the remainder of the set. By some obvious modifications of this example we can build an invertible Peano continuum in the plane with any finite number of orbits. This aspect of the theory is still under investigation so we can not say much more at this time. It is interesting to note, however, that the universal one-dimensional plane curve is an invertible space!

To construct the universal one-dimensional plane curve as we use it, consider a unit square (plus its interior) in the plane. At the first stage, we remove the open square of area 1/9 whose center coincides with the center of the big square

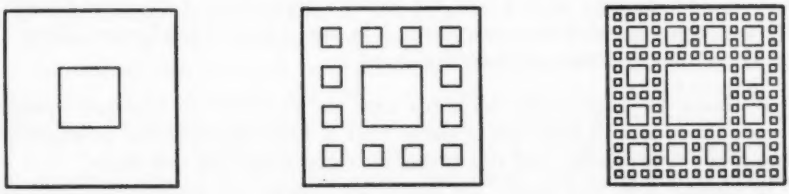

FIG. 2

(Fig. 2-A). At the second stage we remove 12 open squares of area 1/81 as shown in Figure 2-B. The third stage is pictured in Figure 2-C and removes 96 open squares of area $1/3^6$.

To see how this continuum may be inverted, we point out that if we consider

any little square whose interior is removed at the nth stage, then the interior of a square concentric and nine times the area of the removed square contains a set homeomorphic to the complement of the same larger square. By mapping the boundary of the middle square onto the outer boundary of the entire continuum, the remainder is merely a matter of easy matching of the missing squares. This will require a little construction which the reader may supply as a simple puzzle.

Another direction in which our investigations extended was suggested by the well-known fact that the n-sphere is not a product space with nondegenerate factor spaces (even when it is a fibre bundle). The usual proof of this theorem entails some rather complicated apparatus from algebraic topology and we hoped that a simple proof using invertibility might be possible. As a consequence of this desire we were led to consider product spaces. Since the rational points in the plane constitute an invertible product space, we restricted attention to product spaces with compact and connected factors. It would have been nice to have shown that such a product space cannot be invertible. Alas, this hope was rudely dashed when we discovered the following result.

THEOREM 11. *The product of infinitely many closed intervals is invertible.*

Proof. The proof of this theorem depends upon the following lemma in a way that will be obvious if the definition of an open set in the Tychonoff topology of such an infinite product is considered closely.

LEMMA 12. *Let U be an open set in I^n, the product of n intervals. Then there exists a homeomorphism h of I^{n+1} onto itself such that $h(I^{n+1}-U\times I^1)$ lies in $U\times I^1$.*

Proof. In U we choose an $(n-1)$-sphere S^{n-1} (and we mean a geometric sphere). In I^{n+1} consider the $(n+1)$-cell $V\times[0, \frac{1}{2}]$ where V is the n-cell bounded by S^{n-1}. The set $S^{n-1}\times[0, \frac{1}{2}]\cup V\times\frac{1}{2}$ is the intersection of two $(n+1)$-cells in I^{n+1} and the desired homeomorphism h is one which interchanges these two $(n+1)$-cells leaving their intersection fixed.

Theorem 11 discouraged us a bit and we have yet to return to product spaces. There is still hope for a proof that a finite-dimensional product continuum is not invertible but this is merely a conjecture at this stage.

In closing this first report on invertible spaces, we conjecture that other areas of topology may conceal interesting applications of invertibility. Among these might be more results on plane and higher dimensional continua, topological groups and function spaces and certain aspects of homotopy theory. At the time of this writing, for instance, we are enjoying some success in the application of continuous invertibility. (A space S is *continuously invertible* if it is invertible and if an inverting homeomorphism for each open set may be chosen to be isotopic to the identity mapping. The spheres are continuously invertible, for example.) But this is the subject of another report.

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SOME SINGULAR CASES OF THE IMPLICIT FUNCTION THEOREM*

W. S. LOUD, University of Minnesota

1. **Introduction.** The following problem arises in the study of periodic solutions of perturbed autonomous second-order differential equations ([2], [3]).

Let $F(x, y, z)$ and $G(x, y, z)$ have a sufficient number of continuous partial derivatives in some neighborhood of $(0, 0, 0)$. Let $F(0, 0, 0) = G(0, 0, 0) = 0$. It is desired to find x and y as functions of z from

$$F(x, y, z) = 0, \qquad G(x, y, z) = 0, \tag{1.1}$$

where at $z=0$, $x=0$, $y=0$, and the derivatives dx/dz and dy/dz are finite. For the purposes of this paper, finding x and y will mean establishing the existence of x and y as functions of z, and evaluating the derivatives dx/dz and dy/dz at $z=0$. It would be possible in all cases to evaluate higher derivatives of x and y at $z=0$ (or to show they do not exist, in which case higher order behavior could still be studied), but we do not do this.

In the classical case, where the Jacobian of F and G with respect to x and y is nonzero at $(0, 0, 0)$, the standard implicit function gives the existence and uniqueness of x and y as functions of z, and the derivatives in question can be easily computed. However, there are many cases with vanishing Jacobian which can be handled. It is the purpose of this paper to study several of these cases and to show how the existence or nonexistence of the solutions x and y with finite derivatives and $z=0$ can be determined. A complete analysis is extremely complicated. For an example of a complete analysis see [1], pages 163–169. We propose to consider all possible cases that are determinate when the derivatives of F and G through third order are known at $(0, 0, 0)$. In every case then we shall construct the solution in the sense mentioned above, or else prove that there is no solution of the type sought, or else show that the case is not determinate when only the derivatives through order three are known.

We do not wish to be involved with elaborate continuity hypotheses. For the entire paper we make the following agreement. If derivatives of order k but not of order $k+1$ are needed to resolve a situation ($k=1, 2, 3$), then both F and G will be considered in class C^k in a neighborhood of $(0, 0, 0)$.

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2. The classical case. If $F(x, y, z)$ and $G(x, y, z)$ are in C^1 in some neighborhood of $(0, 0, 0)$, and the Jacobian

$$\begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix} \neq 0 \tag{2.1}$$

at $(0, 0, 0)$, the classical implicit function theorem guarantees the existence and uniqueness of the desired functions $x(z)$ and $y(z)$, and of their derivatives at $z=0$. The latter are given by

$$\frac{dx}{dz} = -\frac{\begin{vmatrix} F_z & F_y \\ G_z & G_y \end{vmatrix}}{\begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}}, \qquad \frac{dy}{dz} = -\frac{\begin{vmatrix} F_x & F_z \\ G_x & G_z \end{vmatrix}}{\begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}}, \tag{2.2}$$

the partial derivatives being evaluated at $(0, 0, 0)$.

3. The case of vanishing Jacobian. When the Jacobian (2.1) is zero, the classical implicit function theorem no longer applies. Indeed it is in general not true that functions $x(z)$ and $y(z)$ of the type sought even exist, without some additional assumptions.

In what follows, all partial derivatives of F and G are evaluated at $(0, 0, 0)$, and this fact will not be explicitly noted.

Since the derivatives dx/dz and dy/dz at $z=0$ must satisfy

$$\begin{aligned} F_x \frac{dx}{dz} + F_y \frac{dy}{dz} + F_z &= 0, \\ G_x \frac{dx}{dz} + G_y \frac{dy}{dz} + G_z &= 0, \end{aligned} \tag{3.1}$$

a necessary condition for the existence and finiteness of these derivatives is that the matrix

$$\begin{Vmatrix} F_x & F_y & F_z \\ G_x & G_y & G_z \end{Vmatrix} \tag{3.2}$$

have the same rank as the Jacobian matrix

$$\begin{Vmatrix} F_x & F_y \\ G_x & G_y \end{Vmatrix}. \tag{3.3}$$

If the rank of (3.2) is greater than the rank of (3.3), no solution of the type sought exists (although x and y may sometimes be determined as functions of z with dx/dz and/or dy/dz infinite at $z=0$).

4. Jacobian matrix of rank 1. In this section we assume that the rank of the Jacobian matrix (3.3) is 1. This means that at least one of its entries is nonzero,

and we assume for definiteness that $F_z(0, 0, 0) \neq 0$.

We begin by replacing the system $F(x, y, z) = 0$, $G(x, y, z) = 0$ by a somewhat simpler equivalent system. Define $H(x, y, z)$ by

$$H(x, y, z) \equiv G(x, y, z) - (G_z/F_z)F(x, y, z). \tag{4.1}$$

Because (3.3) has rank 1, and the augmented matrix (3.2) has rank 1 (as we must assume), all three of H_x, H_y, and H_z are zero at (0, 0, 0). We consider the equivalent system

$$F(x, y, z) = 0, \qquad H(x, y, z) = 0. \tag{4.2}$$

The process of solution of the system (4.2) is to eliminate the unknown x from the system. Because $F_x \neq 0$, we can solve the equation $F(x, y, z)$ for x as a function of y and z near $y = z = 0$. If the result is written $x = f(y, z)$, then $f(0, 0) = 0$, and the partial derivatives of $f(y, z)$ at (0, 0) can be computed from the partial derivatives of $F(x, y, z)$ at (0, 0, 0). Now consider the equation

$$J(y, z) \equiv H(f(y, z), y, z) = 0. \tag{4.3}$$

If (4.3) is solved for y as a function of z, $y = y(z)$, with dy/dz finite at $z = 0$, this $y(z)$, together with $x(z) \equiv f(y(z), z)$, furnish the desired solution of the system $F(x, y, z) = 0$, $H(x, y, z) = 0$. Moreover, the derivative dx/dz is given by

$$\frac{dx}{dz} = -\frac{F_y(dy/dz) + F_z}{F_x}. \tag{4.4}$$

Assuming continuity of all derivatives of F and H which appear, the following expressions are readily found for the derivatives of $J(y, z)$ of the first three orders at (0, 0):

$$J_y = 0, \qquad J_z = 0,$$

$$J_{yy} = (H_{xx}F_y^2 - 2H_{xy}F_yF_x + H_{yy}F_x^2)/F_x^2,$$

$$J_{yz} = (H_{xx}F_yF_z - H_{xy}F_zF_x - H_{xz}F_yF_x + H_{yz}F_x^2)/F_x^2,$$

$$J_{zz} = (H_{xx}F_z^2 - 2H_{xz}F_zF_x + H_{zz}F_x^2)/F_x^2,$$

$$\begin{aligned} J_{yyy} = &-(H_{xxx}F_y^3 - 3H_{xxy}F_y^2F_x + 3H_{xyy}F_yF_x^2 - H_{yyy}F_x^3)/F_x^3, \\ &+ 3H_{xx}(F_{xx}F_y^2 - 2F_{xy}F_yF_x + F_{yy}F_x^2)/F_x^4 \\ &- 3H_{xy}(F_{xx}F_y - 2F_{xy}F_yF_x + F_{yy}F_x^2)/F_x^3, \end{aligned}$$

$$\begin{aligned} J_{yyz} = &-(H_{xxx}F_y^2F_z - 2H_{xxy}F_yF_xF_z - H_{xxz}F_y^2F_x + H_{xyy}F_xF_z^2 + 2H_{xyz}F_yF_x^2 \\ &\qquad - H_{yyz}F_x^3)/F_x^3 \\ &+ H_{xx}(3F_{xx}F_y^2F_z - 4F_{xy}F_yF_xF_z - 2F_{xz}F_y^2F_x + F_{yy}F_xF_z^2 + 2F_{yz}F_yF_x^2)/F_x^4 \\ &- 2H_{xy}(F_{xx}F_yF_z - F_{xy}F_xF_z - F_{xz}F_yF_x + F_{yz}F_x^2)/F_x^3 \\ &- H_{xz}(F_{xx}F_y^2 - 2F_{xy}F_yF_x + F_{yy}F_x^2)/F_x^3, \end{aligned}$$

$$J_{yzz} = -(H_{zzz}F_y F_z^2 - H_{zzy}F_z^2 F_z - 2H_{zzz}F_y F_z F_z + 2H_{zyz}F_z F_z^2 + H_{zzz}F_y F_z^2 - H_{yzz}F_z^3)/F_z^4$$
$$+ H_{zz}(3F_{zz}F_y F_z^2 - 2F_{zy}F_z^2 F_z - 4F_{zz}F_y F_z F_z + 2F_{yz}F_z F_z^2 + F_{zz}F_y F_z^2)/F_z^4$$
$$- H_{zy}(F_{zz}F_z^2 - 2F_{zz}F_z F_z + F_{zz}F_z^2)/F_z^3$$
$$- 2H_{zz}(F_{zz}F_y F_z - F_{zy}F_z F_z - F_{zz}F_y F_z + F_{yz}F_z^2)/F_z^3,$$

$$J_{zzz} = -(H_{zzz}F_z^3 - 3H_{zzz}F_z^2 F_z + 3H_{zzz}F_z F_z^2 - H_{zzz}F_z^3)/F_z^3$$
$$+ 3H_{zz}(F_{zz}F_z^3 - 2F_{zz}F_z^2 F_z + F_{zz}F_z F_z^2)/F_z^4$$
$$- 3H_{zz}(F_{zz}F_z^2 - 2F_{zz}F_z F_z + F_{zz}F_z^2)/F_z^3.$$

Since the first partial derivatives of $J(y, z)$ vanish at $(0, 0)$, we can write

$$J(y, z) = \tfrac{1}{2}J_{yy}y^2 + J_{yz}yz + \tfrac{1}{2}J_{zz}z^2 + \tfrac{1}{6}J_{yyy}y^3 + \tfrac{1}{2}J_{yyz}y^2z + \tfrac{1}{2}J_{yzz}yz^2 + \tfrac{1}{6}J_{zzz}z^3 + \text{higher-or}\cdot\text{ terms.} \tag{4.5}$$

We are interested in a solution of $J(y, z)=0$ which has $y=0$ at $z=0$, and for which z takes nonzero values. Write $y=\eta z$ in (4.5). We then obtain

$$J(y, z) = z^2(\tfrac{1}{2}J_{yy}\eta^2 + J_{yz}\eta + \tfrac{1}{2}J_{zz}) + z^3(\tfrac{1}{6}J_{yyy}\eta^3 + \tfrac{1}{2}J_{yyz}\eta^2 + \tfrac{1}{2}J_{yzz}\eta + \tfrac{1}{6}J_{zzz}) + \text{higher-order terms,} \tag{4.6}$$

so that for $z\neq 0$, $J(y, z)=0$ is equivalent to

$$J(\eta, z) = (\tfrac{1}{2}J_{yy}\eta^2 + J_{yz}\eta + \tfrac{1}{2}J_{zz}) + z(\tfrac{1}{6}J_{yyy}\eta^3 + \tfrac{1}{2}J_{yyz}\eta^2 + \tfrac{1}{2}J_{yzz}\eta + \tfrac{1}{6}J_{zzz}) + \text{higher-order terms} = 0. \tag{4.7}$$

We shall solve (4.7) for η as a function of z. The solution of $J(y, z)=0$ for y will then be given by $y(z)=z\eta(z)$.

Consider the quadratic equation

$$\tfrac{1}{2}J_{yy}\eta^2 + J_{yz}\eta + \tfrac{1}{2}J_{zz} = 0, \tag{4.8}$$

which is the limit of (4.7) as $z\to 0$. If $\eta=\eta(z)$ is a solution of (4.7), $\eta(0)$ must satisfy (4.8). There are several possibilities.

(a) If the discriminant $J_{yz}^2 - J_{yy}J_{zz}$ of (4.8) is negative, or if $J_{yy}=J_{yz}=0$ and $J_{zz}\neq 0$, then (4.8) has no real roots. Thus no real value for $\eta(0)$ can exist, so that a solution of the type sought does not exist.

(b) If the discriminant $J_{yz}^2 - J_{yy}J_{zz}$ is positive, (4.8) has one or two simple real roots, according as $J_{yy}=0$ or $J_{yy}\neq 0$. In either case, the ordinary implicit function theorem then guarantees a solution of equation (4.7) $\eta=\eta(z)$ for each such simple real root of (4.8). If η_0 is such a real root, we have

$$\eta(z) = \eta_0 + o(1),$$
$$y(z) = z\eta(z) = \eta_0 z + o(z), \tag{4.9}$$
$$dy/dz = \eta_0 \quad \text{at} \quad z = 0.$$

From (4.4) we find that at $z=0$

$$\frac{dx}{dz} = -\frac{F_y\eta_0 + F_z}{F_x}. \tag{4.10}$$

(c) If the discriminant is zero, and $J_{yy}\neq 0$, we can still obtain $\eta(z)$ for either positive z or negative z in certain cases, but fractional powers of z will be needed. We shall also have to use the third derivative terms. Let η_0 be a double root of (4.8). (4.7) then becomes

$$\begin{aligned}\tfrac{1}{2}J_{yy}(\eta-\eta_0)^2 + z(\tfrac{1}{6}J_{yyy}(\eta-\eta_0)^3 + J_1(\eta-\eta_0)^2 + J_2(\eta-\eta_0) + J_3)\\ + \text{higher-order terms} = 0,\end{aligned} \tag{4.11}$$

where

$$J_1 = \tfrac{1}{2}J_{yyy}\eta_0 + \tfrac{1}{2}J_{yyz}, \qquad J_2 = \tfrac{1}{2}J_{yyy}\eta_0^2 + J_{yyz}\eta_0 + \tfrac{1}{2}J_{yzz},$$
$$J_3 = \tfrac{1}{6}J_{yyy}\eta_0^3 + \tfrac{1}{2}J_{yyz}\eta_0^2 + \tfrac{1}{2}J_{yzz}\eta_0 + \tfrac{1}{6}J_{zzz}.$$

If $J_3=0$, the case can not be resolved without knowledge of higher derivatives, so we do not treat the case.

If $J_3\neq 0$, we replace z by u^2 if J_3 and J_{yy} have opposite signs, and by $-u^2$ if J_3 and J_{yy} have the same sign. Then by taking a square root, we find either

$$\eta - \eta_0 = \pm\sqrt{(-2J_3/J_{yy})}u + \text{higher-order terms}$$

or

$$\eta - \eta_0 = \pm\sqrt{(2J_3/J_{yy})}u + \text{higher-order terms};$$

whence there are two solutions $\eta(z)$ for positive z and none for negative z in the former case, and two solutions $\eta(z)$ for negative z and none for positive z in the latter case. In any case we have $y=\eta_0 z+O(|z|^{3/2})$ so that at $z=0$, $dy/dz=\eta_0$, and dx/dz at $z=0$ is given by (4.10).

(d) If $J_{yy}=J_{yz}=J_{zz}=0$, we can replace (4.6) by

$$\tfrac{1}{6}J_{yyy}\eta^3 + \tfrac{1}{2}J_{yyz}\eta^2 + \tfrac{1}{2}J_{yzz}\eta + \tfrac{1}{6}J_{zzz} + \text{higher-order terms} = 0. \tag{4.12}$$

Consider the equation

$$\tfrac{1}{6}J_{yyy}\eta^3 + \tfrac{1}{2}J_{yyz}\eta^2 + \tfrac{1}{2}J_{yzz}\eta + \tfrac{1}{6}J_{zzz} = 0, \tag{4.13}$$

which is the limit of (4.12) for $z\to 0$. If (4.13) has no real roots, there are no solutions of the type sought. If η_0 is a multiple real root of (4.13), higher derivatives are required to resolve the case, and we do not treat it. Finally, for any simple root, η_0, of (4.13), the ordinary implicit function theorem shows that (4.12) has a solution $\eta=\eta(z)$ with $\eta(z)=\eta_0+o(1)$, so that again $y(z)=\eta_0 z+o(z)$, and at $z=0$,

$$\frac{dy}{dz} = \eta_0, \qquad \frac{dx}{dz} = -\frac{F_y\eta_0 + F_z}{F_x}.$$

Note that in this final case there may be as many as three different solutions $y=y(z)$, $x=x(z)$ of the type sought.

5. Jacobian matrix of rank 0. In the present section we assume that all four of F_x, F_y, G_x, and G_y are zero at (0, 0, 0). Since it is also necessary that the augmented matrix (3.2) have rank zero, we also assume that F_z and G_z are zero at (0, 0, 0). Thus the expansions of $F(x, y, z)$ and $G(x, y, z)$ begin with terms of second degree in (x, y, z).

As in Section 4, we set $x=\xi z$, $y=\eta z$, and substitute in the equations $F(x, y, z)=0$, $G(x, y, z)=0$. On dividing by z^2, we obtain

$$
\begin{aligned}
\bar{F}(\xi, \eta, z) &= \tfrac{1}{2}F_{xx}\xi^2 + F_{xy}\xi\eta + \tfrac{1}{2}F_{yy}\eta^2 + F_{xz}\xi + F_{yz}\eta + \tfrac{1}{2}F_{zz} \\
&\quad + z(\tfrac{1}{6}F_{xxx}\xi^3 + \tfrac{1}{2}F_{xxy}\xi^2\eta + \tfrac{1}{2}F_{xyy}\xi\eta^2 + \tfrac{1}{6}F_{yyy}\eta^3 \\
&\quad + \tfrac{1}{2}F_{xxz}\xi^2 + F_{xyz}\xi\eta + \tfrac{1}{2}F_{yyz}\eta^2 + \tfrac{1}{2}F_{xzz}\xi + \tfrac{1}{2}F_{yzz}\eta + \tfrac{1}{6}F_{zzz}) \\
&\quad + \text{higher-order terms} = 0, \\
\bar{G}(\xi, \eta, z) &\equiv \tfrac{1}{2}G_{xx}\xi^2 + G_{xy}\xi\eta + \tfrac{1}{2}G_{yy}\eta^2 + G_{xz}\xi + G_{yz}\eta + \tfrac{1}{2}G_{zz} \\
&\quad + z(\tfrac{1}{6}G_{xxx}\xi^3 + \tfrac{1}{2}G_{xxy}\xi^2\eta + \tfrac{1}{2}G_{xyy}\xi\eta^2 + \tfrac{1}{6}G_{yyy}\eta^3 \\
&\quad + \tfrac{1}{2}G_{xxz}\xi^2 + G_{xyz}\xi\eta + \tfrac{1}{2}G_{yyz}\eta^2 + \tfrac{1}{2}G_{xzz}\xi + \tfrac{1}{2}G_{yzz}\eta + \tfrac{1}{6}G_{zzz}) \\
&\quad + \text{higher-order terms} = 0.
\end{aligned}
\tag{5.1}
$$

We solve (5.1) for ξ and η as functions of z, and obtain the desired $x(z)$ and $y(z)$ from $x(z)=z\xi(z)$, $y(z)=z\eta(z)$. As in Section 4, we find that at $z=0$, $dx/dz=\xi_0$, $dy/dz=\eta_0$, where ξ_0 and η_0 are the values of ξ and η at $z=0$.

If we let $z\to 0$ in (5.1) we obtain the equations

$$
\begin{aligned}
\bar{F}_0(\xi, \eta) &\equiv \tfrac{1}{2}F_{xx}\xi^2 + F_{xy}\xi\eta + \tfrac{1}{2}F_{yy}\eta^2 + F_{xz}\xi + F_{yz}\eta + \tfrac{1}{2}F_{zz} = 0, \\
\bar{G}_0(\xi, \eta) &\equiv \tfrac{1}{2}G_{xx}\xi^2 + G_{xy}\xi\eta + \tfrac{1}{2}G_{yy}\eta^2 + G_{xz}\xi + G_{yz}\eta + \tfrac{1}{2}G_{zz} = 0.
\end{aligned}
\tag{5.2}
$$

The possible values of (ξ_0, η_0) are solutions of the system (5.2).

There are two degenerate cases which we discuss first. If both of $\bar{F}_0(\xi, \eta)$ and $\bar{G}_0(\xi, \eta)$ vanish identically, we must find ξ and η from the pair of equations

$$
\begin{aligned}
&\tfrac{1}{6}F_{xxx}\xi^3 + \tfrac{1}{2}F_{xxy}\xi^2\eta + \tfrac{1}{2}F_{xyy}\xi\eta^2 + \tfrac{1}{6}F_{yyy}\eta^3 + \tfrac{1}{2}F_{xxz}\xi^2 + F_{xyz}\xi\eta \\
&\quad + \tfrac{1}{2}F_{yyz}\eta^2 + \tfrac{1}{2}F_{xzz}\xi + \tfrac{1}{2}F_{yzz}\eta + \tfrac{1}{6}F_{zzz} + \text{higher-order terms} = 0, \\
&\tfrac{1}{6}G_{xxx}\xi^3 + \tfrac{1}{2}G_{xxy}\xi^2\eta + \tfrac{1}{2}G_{xyy}\xi\eta^2 + \tfrac{1}{6}G_{yyy}\eta^3 + \tfrac{1}{2}G_{xxz}\xi^2 + G_{xyz}\xi\eta \\
&\quad + \tfrac{1}{2}G_{yyz}\eta^2 + \tfrac{1}{2}G_{xzz}\xi + \tfrac{1}{2}G_{yzz}\eta + \tfrac{1}{6}G_{zzz} + \text{higher-order terms} = 0.
\end{aligned}
\tag{5.3}
$$

In this case, (ξ_0, η_0) is a solution of the pair of equations obtained from (5.3) by setting the higher order terms equal to zero.

If this reduced system has no real solutions, no solutions of the type sought exist for the problem. If (ξ_0, η_0) is a multiple root of this system, higher derivatives F and G are needed to resolve the situation, and we do not treat it. For any simple root (ξ_0, η_0) of the reduced system, the ordinary implicit function

theorem guarantees the existence of a solution $(\xi(z), \eta(z))$ of (5.3) with $\xi(0)=\xi_0$ and $\eta(0)=\eta_0$. Hence we obtain for each such simple root (ξ_0, η_0) a solution $(x(z), y(z))$ of the original problem with $dx/dz=\xi_0$, $dy/dz=\eta_0$ at $z=0$. There is the possibility of as many as nine different solutions occurring in this case.

(Note: A root (ξ_0, η_0) is a simple root of the reduced system if the Jacobian of the two equations is nonzero at (ξ_0, η_0).)

The second degenerate case is that in which one but not both of $\bar{F}_0(\xi, \eta)$ and $\bar{G}_0(\xi, \eta)$ vanish identically, or neither vanishes identically, but one is a constant multiple of the other. We assume for definiteness that $\bar{F}_0(\xi, \eta) \not\equiv 0$. Then by subtraction of a suitable multiple of $F(x, y, z)$ from $G(x, y, z)$, we can transform the system so that $\bar{G}_0(\xi, \eta) \equiv 0$. We find ξ_0 and η_0 in this case from the system formed from $\bar{F}_0(\xi, \eta)=0$ and the second equation of (5.3) with higher order terms set equal to zero.

Exactly the same remarks now hold for this system of two equations. If it has no real solutions, no solution $(x(z), y(z))$ of the form sought exists. Multiple roots bring us to a case that can not be resolved without knowing higher derivatives. For each simple real root (ξ_0, η_0) there is a solution $(x(z), y(z))$ of the original system with $dx/dz=\xi_0$, $dy/dz=\eta_0$ at $z=0$. There may be as many as six such solutions in this case.

Having disposed of these degenerate cases, we now return to the system (5.2), where we assume that neither of $\bar{F}_0(\xi, \eta)$ and $\bar{G}_0(\xi, \eta)$ vanishes identically, and that they are not proportional. The following possibilities exist for intersections of the loci $\bar{F}_0(\xi, \eta)=0$ and $\bar{G}_0(\xi, \eta)=0$.

There may be no real intersections. In this case there will be no solution $(x(z), y(z))$ of the original system of the type sought. There may be a finite number of isolated intersections, at most four in number. We shall analyze each possibility here later.

The two loci may be both degenerate and have a single straight line in common. If the loci have more than a single line in common, they must coincide, which brings us to one of the degenerate situations already treated.

We consider the case of an isolated intersection first. If (ξ_0, η_0) is an isolated solution of the system (5.2), we make the substitution $\xi-\xi_0=u$, $\eta-\eta_0=v$. The system of equations (5.1) then takes the form

$$
\begin{aligned}
&F_1u + F_2v + \tfrac{1}{2}F_{xx}u^2 + F_{xy}uv + \tfrac{1}{2}F_{yy}v^2 + z(F_3 + F_4u + F_5v + \cdots \\
&\quad + \tfrac{1}{6}F_{xxx}u^3 + \cdots + \tfrac{1}{6}F_{yyy}v^3) + \text{higher-order terms} = 0, \\
&G_1u + G_2v + \tfrac{1}{2}G_{xx}u^2 + G_{xy}uv + \tfrac{1}{2}G_{yy}v^2 + z(G_3 + G_4u + G_5v + \cdots \\
&\quad + \tfrac{1}{6}G_{xxx}u^3 + \cdots + \tfrac{1}{6}G_{yyy}v^3) + \text{higher-order terms} = 0.
\end{aligned} \tag{5.4}
$$

In (5.4) the coefficients F_1, F_2, F_3, G_1, G_2, G_3, etc. are evaluated from the various partial derivatives of $\bar{F}_0(\xi, \eta)$ and $\bar{G}_0(\xi, \eta)$ evaluated at (ξ_0, η_0).

We must merely establish that (5.4) has a solution $(u(z), v(z))$ near $z=0$ with $u(0)=v(0)=0$, of some form. Once this is done we have successively

$$\xi(z) = \xi_0 + u(z), \qquad \eta(z) = \eta_0 + v(z);$$
$$x(z) = \xi_0 z + zu(z), \qquad y(z) = \eta_0 z + zv(z);$$

so that $x(z)$ and $y(z)$ of the form sought exist, and at $z=0$, $dx/dz=\xi_0$, $dy/dz=\eta_0$.

If the Jacobian of the system (5.4) with respect to u and v,

$$\begin{vmatrix} F_1 & F_2 \\ G_1 & G_2 \end{vmatrix},$$

is nonzero, the standard implicit function theorem asserts the existence of the solution $(u(z), v(z))$ of (5.4), and thus the existence of the solution $x(z)$, $y(z)$ of the original system in the form sought.

If the above Jacobian is zero, we may sometimes obtain the desired solution. Suppose first that the rank of the matrix

$$\left\| \begin{matrix} F_1 & F_2 \\ G_1 & G_2 \end{matrix} \right\| \tag{5.5}$$

is 1, and assume for definiteness that $F_1 \neq 0$. If the rank of the augmented matrix

$$\left\| \begin{matrix} F_1 & F_2 & F_3 \\ G_1 & G_2 & G_3 \end{matrix} \right\| \tag{5.6}$$

is also 1, we are in a situation similar to that in Section 4, and higher order derivatives are necessary to resolve the situation, so we do not handle the case. On the other hand, if the rank of (5.6) is 2, we can proceed.

Since $F_1 \neq 0$, solve the first equation of (5.4) for u as a function of v and z, and substitute in the second equation. The result is

$$(F_1G_3 - F_3G_1)z + \frac{(G_{zz}F_1 - F_{zz}G_1)F_2^2 - 2(G_{zy}F_1 - F_{zy}G_1)F_1F_2 + (G_{yy}F_1 - F_{yy}G_1)F_1^2}{2F_1^2}v^2 \tag{5.7}$$
$$+ \text{ terms in } vz \text{ and } z^2 + \text{higher-order terms} = 0.$$

The coefficient of z in (5.7) is not zero because the rank of (5.6) is 2. If the coefficient of v^2 is zero, the situation cannot be resolved without the use of higher derivatives, so we do not treat this case. If the coefficient of v^2 is nonzero, we are in a situation similar to (c) in Section 4. If the coefficients of z and v^2 have opposite signs, (5.7) has two solutions $v(z)$ for positive z, and none for negative z. If these coefficients have the same sign, (5.7) has two solutions $v(z)$ for negative z, and none for positive z. Thus again we have two solutions $(x(z), y(z))$ with derivatives at $z=0$ given by ξ_0 and η_0 respectively. The solutions exist for positive z and not for negative z if the coefficients of z and v^2 in (5.7) have opposite signs, and for negative z and not for positive z if these coefficients have the same sign.

Now suppose that the matrix (5.5) has rank zero. If (5.6) also has rank zero, higher derivatives than third of F and G are needed to resolve the situation, so

we do not treat it. If, on the other hand, the rank of (5.6) is one, we can proceed.

Assume for definiteness that $F_3 \neq 0$. Then the first equation of (5.4) may be solved for z as a function of u and v.

$$z = -\tfrac{1}{2}(F_{xx}/F_3)u^2 - (F_{xy}/F_3)uv - \tfrac{1}{2}(F_{yy}/F_3)v^2 + \text{higher-order terms.} \tag{5.8}$$

If (5.8) is then substituted into the second equation of (5.4), an equation involving u and v only is obtained

$$\tfrac{1}{2}(F_{xx}G_3 - G_{xx}F_3)u^2 + (F_{xy}G_3 - G_{xy}F_3)uv + \tfrac{1}{2}(F_{yy}G_3 - G_{yy}F_3)v^2 + \text{higher-order terms} = 0. \tag{5.9}$$

Not all three of the coefficients of u^2, uv, and v^2 are zero in (5.9), for in that case, $F_0(\xi_0, \eta_0)$ and $G_0(\xi_0, \eta_0)$ would be proportional, or else one would vanish identically, and we would have a degenerate case. Consider the discriminant

$$(F_{xy}G_3 - G_{xy}F_3)^2 - (F_{xx}G_3 - G_{xx}F_3)(F_{yy}G_3 - G_{yy}F_3). \tag{5.10}$$

If (5.10) is negative, the only real point near (0, 0, 0) common to the loci determined by (5.4) is $u=v=z=0$, so no solution of the type sought exists. If (5.10) is zero, higher derivatives are required to resolve the situation, so we do not treat the case. If (5.10) is positive, the locus of the equation formed from (5.9) by omitting higher order terms consists of two distinct straight lines. Moreover, if $au+bv=0$ is such a straight line, $au+bv$ is not a factor of $\frac{1}{2}F_{xx}u^2+F_{xy}uv+\frac{1}{2}F_{yy}v^2$, for if it were, $u=0, v=0$, which is the same as $\xi=\xi_0, \eta=\eta_0$ would not be an isolated intersection of the curves defined by (5.4). As a result for each of the two lines, we can solve (5.9) for u as a function of v, or v as a function of u (perhaps both), and substitute the result in (5.8). The result will have the form

$$z = ku^2 + \text{higher order terms} \quad \text{or} \quad z = kv^2 + \text{higher-order terms,}$$

with $k \neq 0$ in either case. Therefore each of the above two straight lines gives rise to two solutions $(u(z), v(z))$ of the system (5.4), both valid for positive z only if the corresponding k is positive, and both valid for negative z only if the corresponding k is negative. Hence in the case that (5.10) is positive, the original problem has four solutions of the type sought, of which four, two, or none are valid for positive z, with the remaining ones valid for negative z. For all four $dx/dz=\xi_0$ and $dy/dz=\eta_0$ at $z=0$.

We now consider the final situation, that in which the two loci $\bar{F}_0(\xi, \eta)=0$ and $\bar{G}_0(\xi, \eta)=0$ of (5.4) have a single straight line in common. We let (ξ_0, η_0) be any point on the common line, and make the substitution $u=\xi-\xi_0, v=\eta-\eta_0$. The system (5.4) then takes the form

$$\begin{aligned}(Au + Bv)(A_1u + B_1v + C_1) + z(F_3 + F_4u + F_5v + \cdots) &= 0,\\ (Au + Bv)(A_2u + B_2v + C_2) + z(G_3 + G_4u + G_5v + \cdots) &= 0,\end{aligned} \tag{5.11}$$

where A and B are not both zero, and to avoid the degenerate cases mentioned earlier, the matrix

$$\begin{Vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{Vmatrix}$$

has rank 2. This rank is independent of the choice of the point (ξ_0, η_0) on the common line. Certain of the conditions that will appear later will be satisfied only for certain points on the common line.

Suppose first that not both of C_1 and C_2 are zero. Assume for definiteness that $C_1 \neq 0$. Since not both A and B are zero, we assume without loss of generality that $A \neq 0$. We may then solve the first of equations (5.11) for u as a function of v and z,

$$u = -\frac{B}{A}v - \frac{F_3}{AC_1}z + \frac{F_3(AB_1 - A_1B) - C_1(AF_5 - BF_4)}{A^2C_1^2}vz + \text{terms in } z^2 + \text{higher-order terms}. \tag{5.12}$$

If (5.12) is substituted into the second equation of (5.11), the result is

$$(C_1G_3 - C_2F_3)z + \frac{C_2F_3(AB_1 - BA_1) - C_1F_3(AB_2 - BA_2) - C_1C_2(AF_5 - BF_4) + C_1^2(AG_5 - BG_4)}{AC_1}vz + \text{terms in } z^2 + \text{higher-order terms} = 0. \tag{5.13}$$

The left member of (5.13) is divisible by z. If we divide by z, we see that (5.13) has no solution $v = v(z)$ with $v(0) = 0$ unless $C_1G_3 - C_2F_3 = 0$. (Here is the condition that determines which points (ξ_0, η_0) on the common line must be chosen.) If $C_1G_3 - C_2F_3 = 0$, and the coefficient of vz in (5.13) is not zero, then (5.13) has a solution $v = v(z)$ with $v(0) = 0$, and using (5.12) we can also find $u(z)$. In this case then a solution of the type sought exists. If the coefficient of vz in (5.13) is zero, higher derivatives of F and G are needed to resolve the situation, and we do not handle the case.

Now suppose that both C_1 and C_2 are zero. The system (5.11) becomes

$$\begin{aligned} (Au + Bv)(A_1u + B_1v) + z(F_3 + F_4u + F_5v + \cdots) &= 0, \\ (Au + Bv)(A_2u + B_2v) + z(G_3 + G_4u + G_5v + \cdots) &= 0, \end{aligned} \tag{5.14}$$

where A and B are not both zero, and $A_1B_2 - A_2B_1 \neq 0$. If both of F_3 and G_3 are zero, higher derivatives are required to resolve the situation, and we do not handle the case. If not both of F_3 and G_3 are zero, assume for definiteness that $F_3 \neq 0$. Then the first equation of (5.14) can be solved for z as a function of u and v

$$z = -\{(Au + Bv)(A_1u + B_1v)\}/F_3 + \text{higher-order terms}. \tag{5.15}$$

If (5.15) is substituted into the second equation of (5.14), we obtain

$$(Au + Bv)((A_1G_3 - A_2F_3)u + (B_1G_3 - B_2F_3)v) + \text{higher-order terms} = 0. \tag{5.16}$$

Now not both of $A_1G_3-A_2F_3$ and $B_1G_3-B_2F_3$ are zero, since $A_1B_2-A_2B_1\neq 0$, and $F_3\neq 0$. Thus the locus of the equation obtained by dropping the higher order terms in (5.16) consists either of two distinct straight lines or two coincident straight lines. The lines are coincident if and only if

$$\begin{vmatrix} A & B & 0 \\ A_1 & B_1 & F_3 \\ A_2 & B_2 & G_3 \end{vmatrix} = 0. \tag{5.17}$$

If the two lines are coincident, the situation can not be resolved without higher derivatives, and we do not handle it. If the determinant in (5.17) is not zero, the lines are distinct. The line $Au+Bv=0$ leads to a solution $u=u(v)$, or $v=v(u)$ of (5.16) which will require higher derivatives to handle when it is substituted into (5.15), so we do not handle it. The second line

$$(A_1G_3 - A_2F_3)u + (B_1G_3 - B_2F_3)v = 0$$

leads to a situation such as that handled when (5.10) is positive. In this case we obtain two solutions $(u(z), v(z))$ of the system (5.14), valid either for positive z or for negative z but not both.

6. Examples. In this section we give three examples to illustrate the more complicated situations of Section 5.

Example I.

$$x^2 + y^2 - z^2 + z^2x = 0, \qquad x^2 + 2y^2 - z^2 + z^2y = 0.$$

Here the rank of the Jacobian matrix is zero, since there are no linear terms. Setting $x=\xi z$ and $y=\eta z$, we obtain

$$\xi^2 + \eta^2 - 1 + z\xi = 0, \qquad \xi^2 + 2\eta^2 - 1 + z\eta = 0.$$

The real common points of the loci

$$\xi^2 + \eta^2 - 1 = 0, \qquad \xi^2 + 2\eta^2 - 1 = 0$$

are $(1, 0)$ and $(-1, 0)$. At $(1, 0)$, let $u=\xi-1$, $v=\eta$. We get for the system (5.4)

$$2u + u^2 + v^2 + z(u + 1) = 0, \qquad 2u + u^2 + 2v^2 + zv = 0.$$

The matrix (5.6) becomes

$$\begin{Vmatrix} 2 & 0 & 1 \\ 2 & 0 & 0 \end{Vmatrix},$$

which has rank 2. The first equation when solved for u gives

$$u = -\tfrac{1}{2}z - \tfrac{1}{2}v^2 + \tfrac{1}{8}z^2 + \text{higher-order terms.}$$

When substituted into the second equation, this gives in turn

$$-z + v^2 + vz + \tfrac{1}{2}z^2 + \text{higher-order terms} = 0.$$

From this we see that for small positive z, we can write $v = \pm\sqrt{z} + o(\sqrt{z})$. Then in turn we get

$$u = o(\sqrt{z}), \qquad \xi = 1 + o(\sqrt{z}), \qquad \eta = \pm\sqrt{z} + o(\sqrt{z}),$$
$$x = z + o(z^{3/2}), \qquad y = \pm z^{3/2} + o(z^{3/2}),$$

all for small positive z. The point $\xi = -1$, $\eta = 0$ is similar.

Example II.

$$xy + xz - yz - z^2 + z^2x = 0, \qquad x^2 - y^2 - 2xz - 2yz + z^2y = 0.$$

When we set $x = \xi z$ and $y = \eta z$, these become

$$\xi\eta + \xi - \eta - 1 + z\xi = 0, \qquad \xi^2 - \eta^2 - 2\xi - 2\eta + z\eta = 0.$$

The only common point of the loci

$$\xi\eta + \xi - \eta - 1 = 0, \qquad \xi^2 - \eta^2 - 2\xi - 2\eta = 0$$

is $(1, -1)$. Setting $u = \xi - 1$, $v = \eta + 1$, we get for the system (5.4)

$$uv + z(u + 1) = 0, \qquad u^2 - v^2 + z(v - 1) = 0.$$

Solving the first equation for z as a function of u and v, we find

$$z = -uv + \text{higher-order terms}.$$

Substituting in the second equation, we have

$$u^2 + uv - v^2 + \text{higher-order terms} = 0.$$

The discriminant (5.10) is positive, and we can solve the last for v as a function of u:

$$v = \rho u + \cdots \quad \text{and} \quad v = -\rho^{-1}u + \cdots,$$

where $\rho = \frac{1}{2}(1 + \sqrt{5}) \approx 1.618$. Hence we find

$$z = -\rho u^2 + \cdots \quad \text{and} \quad z = \rho^{-1}u^2 + \cdots.$$

The first of these equations gives $u(z)$ for small negative z, while the second gives $u(z)$ for small positive z. We therefore have four solutions for this example, two for small positive z and two for small negative z:

$z > 0$	$z < 0$
$u = \pm\sqrt{(\rho z)} + o(\sqrt{z})$,	$u = \pm\sqrt{(-\rho^{-1}z)} + o(\sqrt{\lvert z\rvert})$,
$v = \pm\sqrt{(\rho^{-1}z)} + o(\sqrt{z})$,	$v = \pm\sqrt{(-\rho z)} + o(\sqrt{\lvert z\rvert})$,
$\xi = 1 \pm\sqrt{(\rho z)} + o(\sqrt{z})$,	$\xi = 1 \pm\sqrt{(-\rho^{-1}z)} + o(\sqrt{\lvert z\rvert})$,
$\eta = -1 \pm\sqrt{(\rho^{-1}z)} + o(\sqrt{z})$,	$\eta = -1 \pm\sqrt{(-\rho z)} + o(\sqrt{\lvert z\rvert})$,

$$x = z \pm \sqrt{(\rho z^3)} + o(z^{3/2}), \qquad x = z \pm \sqrt{(-\rho^{-1}z^3)} + o(|z|^{3/2}),$$
$$y = -z \pm \sqrt{(\rho^{-1}z^3)} + o(z^{3/2}). \qquad y = -z \pm \sqrt{(-\rho z^3)} + o(|z|^{3/2}).$$

Example III.

$$xy - xz + yz^2 = 0, \quad x^2 + xy + y^2z = 0.$$

Setting $x=\xi z$, $y=\eta z$, we obtain

$$\xi\eta - \xi + z\eta = 0, \quad \xi^2 + \xi\eta + z\eta^2 = 0.$$

The loci

$$\xi\eta - \xi = 0, \quad \xi^2 + \xi\eta = 0$$

have the entire line $\xi=0$ in common. (They also have the point $(-1, 1)$ in common, but we do not consider this here.) Select any point $(0, \eta_0)$ on the line $\xi=0$, and set $u=\xi$, $v=\eta-\eta_0$. We obtain

$$u(v + \eta_0 - 1) + z(v + \eta_0) = 0, \quad u(u + v + \eta_0) + z(\eta_0^2 + 2\eta_0 v + v^2) = 0.$$

We find that $C_1=\eta_0-1$, $C_2=\eta_0$, $F_2=\eta_0$, $G_2=\eta_0^2$, so that the critical quantity $C_1G_2-C_2F_2=\eta_0^3-2\eta_0^2$, which is zero only for $\eta_0=0$ and $\eta_0=2$. No other values of η_0 will give solutions of the type sought.

When $\eta_0=0$, we have

$$u(v - 1) + zv = 0, \quad u(u + v) + zv^2 = 0.$$

When the first of these equations is solved for u in terms of v and z and the result is substituted in the second, the coefficient of vz turns out to be zero, so that the methods in the paper do not resolve the situation. (Actually it is clear that $x=0$, $y=0$ is a solution with $\eta_0=0$.)

When $\eta_0=2$, we have

$$u(v + 1) + z(v + 2) = 0, \quad u(u + v + 2) + z(v^2 + 4v + 4) = 0.$$

This time the coefficient of zv turns out to be nonzero, so that we do have a solution $x=o(z)$, $y=2z+o(z)$. This can be verified, since the system can be solved explicitly.

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A NEW GENERALIZATION OF JENSEN'S THEOREM ON THE ZEROS OF THE DERIVATIVE OF A POLYNOMIAL*

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In the geometry of the zeros of the derivative of a polynomial, a central role is taken by Lucas's theorem: *If $p(z)$ is a polynomial not identically constant, in the complex plane the zeros of the derivative $p'(z)$ lie in the smallest convex polygon containing the zeros of the original polynomial.* Under certain conditions Lucas's theorem can be sharpened:

JENSEN'S THEOREM. *Let $p(z)$ be a real polynomial (i.e., with real coefficients) not identically constant, and consider the circles having as diameters the line segments $z_k\bar{z}_k$ joining conjugate pairs of nonreal zeros of $p(z)$. Then all nonreal zeros of $p'(z)$ lie in the closed interiors of these circles.*

The object of the present note is to establish generalizations (Theorems 2 and 3 below) of the following theorem, recently proved [1] by the writer, which is itself a generalization of Jensen's Theorem:

THEOREM 1. *Let $p(z)$ be a real polynomial not identically constant all of whose zeros have real parts in the interval $\alpha \leqq x \leqq \beta$ of the axis of reals, and let γ be a real point not interior to that interval and not having an abscissa equal to that of a nonreal zero of $p(z)$. Let $\Gamma(z_k)$ denote the circle through the conjugate pair $(z_k, \bar{z}_k)$ of nonreal zeros tangent to the line γz_k at z_k. Then all nonreal zeros of the derivative $p'(z)$ lie in the closed interiors of the circles $\Gamma(z_k)$.*

All the theorems mentioned are conveniently proved by use of a suitable field of force ([2], Sec. 4.1.1):

BÔCHER'S THEOREM. *The finite zeros of the derivative $r'(z)$ of a nonconstant rational function $r(z)$ which are not multiple zeros of $r(z)$ are the positions of equilibrium in the field of force due to particles of positive mass at the zeros of $r(z)$ and particles of negative mass at the poles of $r(z)$, with masses numerically equal to the respective multiplicities, where each particle repels with a force equal to the mass times the inverse distance.*

The special case here where $r(z)$ is a polynomial in z is due to Gauss; the single pole of $r(z)$ is then at infinity, and the particle there can be ignored in setting up the field of force. In Bôcher's theorem a particular convention which does not concern us here may be made regarding $z = \infty$ as a zero of $r'(z)$.

Theorem 1 was proved, and succeeding theorems are to be proved, by application of a lemma regarding Bôcher's field of force. The following represents a slight sharpening of the original form [1]:

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LEMMA. *Let Γ denote the circle through the points $+i$ and $-i$, tangent at $+i$ to the line joining $+i$ and $\gamma(>0)$. At an arbitrary nonreal point $z_0=x_0+iy_0$, $y_0>0$, exterior [interior] to Γ the force due to positive unit particles at $+i$ and $-i$ has a nonvanishing component perpendicular to the line γz_0 in the clockwise [counterclockwise] sense about γ.*

For our purposes it is sufficient to study the field of force in the upper half-plane, since considerations of symmetry then yield corresponding results in the lower half-plane. Our main result on rational functions is now available:

THEOREM 2. *Let $r(z)$ be a real rational function not identically constant all of whose finite zeros lie in the half-plane $x>0$ and all of whose finite poles lie in the half-plane $x<0$, except that $z=0$ may be a zero or a pole. Let a circle $\Gamma(z_k)$ be drawn with center on the axis of reals passing through each conjugate pair $(z_k, \bar{z}_k)$ of zeros and of poles, where $\Gamma(z_k)$ is tangent at z_k to the line Oz_k. Then all nonreal zeros of $r'(z)$ lie in the closed interiors of the $\Gamma(z_k)$.*

At an arbitrary point z_0 exterior to the $\Gamma(z_k)$ and in the upper half-plane, the force due to the particles (if any) at the finite zeros of $r(z)$ in $x>0$ has a nonvanishing component perpendicular to the line Oz_0 in the counterclockwise sense; this statement is true by the lemma for the nonreal pairs of particles, and is clearly true so far as concerns the particles on the axis of reals. By the same reasoning, the force at z_0 due to the particles (if any) at the finite poles of $r(z)$ in $x<0$ also has a nonvanishing component perpendicular to the line Oz_0 in the counterclockwise sense. If $r(z)$ is z^m, where m is a positive or negative integer, there are no nonreal zeros of $r'(z)$ and the conclusion of the theorem is satisfied; in any other case there exist either finite zeros of $r(z)$ in $x>0$ or finite poles in $x<0$, or both, and the force at z_0 due to a positive or negative particle at O acts along Oz_0, so z_0 is not a position of equilibrium. Of course z_0 cannot be a multiple zero of $r(z)$, so the theorem follows.

A second theorem due to Bôcher ([2], Sec. 4.2) asserts that *if a line or circle L does not pass through all the zeros and poles of a nonconstant rational function $r(z)$ but separates the zeros of $r(z)$ not on L from the poles of $r(z)$ not on L, then L passes through no finite zero of $r'(z)$ which is not a multiple zero of $r(z)$.* It is to be noted that Theorem 2 is in part concerned with lines L through O which separate the zeros of $r(z)$ not on L from the poles of $r(z)$ not on L; for instance, the axis of imaginaries is such a line L; but if $r(z)$ has nonreal zeros or poles, Theorem 2 is stronger (*i.e.*, for real $r(z)$) than the theorem just quoted.

As a limiting case of the lemma, which may be proved by the original method, we may choose $\gamma=0$; the conclusion is then that *the force at the nonreal point $z_0=x_0+iy_0$, $x_0>0$, $y_0>0$, has a nonvanishing component perpendicular to the line Oz_0 in the clockwise sense.* Consequently, as a limiting case of Theorem 2 we may allow either zeros or poles of $r(z)$ (but not both) to lie on Oy; the modified conclusion in these respective cases is that all nonreal zeros of $r'(z)$ in $x<0$ or in $x>0$ lie in the closed interiors of the Γ_k; in either case no nonreal zero of

$r'(z)$ not a multiple zero of $r(z)$ lies on Oy unless all zeros and poles of $r(z)$ lie on Oy.

An interesting special case of Theorem 2 is that in which $r(z)$ has but a single pole in the extended plane. If this pole is at infinity, Theorem 2 reduces essentially to Theorem 1. If this pole is at a point $z=a(<0)$, and if we transform the plane by a linear transformation of the complex variable which carries the three points a, 0, ∞, into points ∞, α, β, where α and β are real, we obtain a new result concerning polynomials:

THEOREM 3. *Let $p(z)$ be a real polynomial all of whose zeros lie in the closed interior of the circle whose diameter is the segment $\alpha\beta$ of the axis of reals, and let all the nonreal zeros lie interior to that circle. For each pair $(z_k, \bar{z}_k)$ of conjugate imaginary zeros of $p(z)$ let Γ_k be the circle tangent at z_k and $\bar{z}_k$ to the respective circles $\alpha z_k\beta$ and $\alpha\bar{z}_k\beta$. Then all nonreal zeros of $p'(z)$ lie in the closed interiors of the Γ_k. A nonreal point z_0 on a circumference Γ_k but not a multiple zero of $p(z)$ and not on or within a second circle Γ_j cannot be a zero of $p'(z)$ unless $p(z)$ has precisely three distinct zeros, one at α or β.*

In Theorem 3 the limiting cases $\alpha=\infty$, $\beta=\infty$ are not excluded. If we choose $\alpha=\infty$ or $\beta=\infty$ but not both, Theorem 3 reduces to Theorem 1; if we choose $\alpha=\infty$, $\beta=\infty$, Theorem 3 reduces to Jensen's theorem.

The last sentence of Theorem 3 deserves further discussion; the corresponding fact was not mentioned in connection with Theorems 1 and 2. Under the conditions of the Lemma, the line of action of the force at a point z_0 of Γ passes through γ. Thus under the conditions of Theorem 2, if the nonreal point z_0 lies on the circumference $\Gamma(z_1)$, the force at z_0 due to the two particles z_1 and $\bar{z}_1$ has the line of action Oz_0. If there exist other (finite) particles than z_1, $\bar{z}_1$, and O, and if z_0 is not on or within a second circle $\Gamma(z_k)$, the total force at z_0 has a nonzero component perpendicular to Oz_0; so z_0 is not a position of equilibrium. That is to say, *under the conditions of Theorem 2, a nonreal point z_0 on a circumference $\Gamma(z_k)$ but not a multiple zero of $r(z)$ and not on or within a second circumference $\Gamma(z_j)$ cannot be a zero of $r'(z)$ unless $r(z)$ has in the plane of finite points a totality of precisely three distinct zeros and poles, one of which lies at O.*

To discuss the situation just excepted, suppose $z_1=x_1+iy_1$ to be a zero of $r(z)$, $x_1>0$, $y_1>0$, and suppose the nonreal point z_0 to lie on the circumference $\Gamma(z_1)$, on the left-hand arc bounded by z_1 and $\bar{z}_1$. If the algebraic magnitudes of the force at z_0 due to the pair of unit particles z_1 and $\bar{z}_1$ and the force at z_0 due to a unit particle at O are in the ratio $m_2:m_1$, where $m_1m_2<0$, then z_0 satisfies

$$\frac{m_2}{z}-\frac{m_1}{z-z_1}-\frac{m_1}{z-\bar{z}_1}=0,$$

and z_0 is a zero of $r'(z)$ if and only if

$$r(z)\equiv z^{m_2}[(z-z_1)(z-\bar{z}_1)]^{-m_1}, \tag{1}$$

where m_1 and m_2 are not uniquely determined but their ratio is, and we may choose either $m_1>0$, $m_2<0$, or $m_1<0$, $m_2>0$. The function $r(z)$ can here be chosen a rational function, indeed a polynomial, if and only if the ratio $m_2:m_1$ is rational; in the contrary case, z_0 is not the zero of the derivative of any admissible rational function. If z_0 lies on this same circumference $\Gamma(z_1)$, on the right-hand arc bounded by z_1 and $\bar{z}_1$, and if the algebraic magnitudes of the forces as already defined are in the ratio $m_2:m_1$, where $m_1m_2>0$, then z_0 is a zero of $r'(z)$ if and only if $r(z)$ is defined by (1). Again, $r(z)$ can be chosen a rational function if and only if the ratio $m_2:m_1$ is rational and we may choose $m_1>0$, $m_2>0$ or $m_1<0$, $m_2<0$; in the case that $m_2:m_1$ is irrational, z_0 is not the zero of the derivative of any admissible rational function. Illustrations here are

$$r(z) \equiv z(z-1+i)(z-1-i), \qquad r(z) \equiv (z-1+i)(z-1-i)/z^3,$$

whose derivatives have respectively the zeros $\frac{1}{3}(2\pm i\sqrt{2})$ and $2\pm i\sqrt{2}$.

To recapitulate: if the real rational function $r(z)$ has precisely three distinct finite zeros and poles, one of which lies at O and the others are both zeros or both poles, then all nonreal zeros of $r'(z)$ lie on the circle $\Gamma(z_1)$ of Theorem 2; the numerical illustrations just given apply.

Theorem 2 is also of interest in case $r(z)$ has in the plane of finite points only four zeros and poles, which occur in pairs of conjugate nonreal points. If there exists a nonreal zero z_0 of $r'(z)$ not a multiple zero of $r(z)$, the force at z_0 due to one conjugate pair $(z_1, \bar{z}_1)$ is not zero, and its line of action cuts the axis of reals in some point γ, which may be the point at infinity. The line of action of the force at z_0 due to the other conjugate pair also passes through γ. The circle through the points z_k and $\bar{z}_k$ tangent at z_k to the line γz_k passes through z_0 for $k=1$ and $k=2$, so z_0 lies at the intersection of these two circles. In the reciprocal direction, suppose two pairs $(z_k, \bar{z}_k)$ of conjugate points are given, $k=1, 2$. Let γ be real (perhaps infinite), and suppose the two circles just described intersect in a nonreal point z_0. If the algebraic magnitudes of the forces at z_0 due to the respective pairs of particles ($k=1, 2$) are in the ratio $m_2:m_1$, then z_0 satisfies

$$\frac{m_1}{z-z_1}+\frac{m_1}{z-\bar{z}_1}-\frac{m_2}{z-z_2}-\frac{m_2}{z-\bar{z}_2}=0$$

and z_0 is a zero of $r'(z)$ if we have

$$r(z) \equiv [(z-z_1)(z-\bar{z}_1)]^{m_1}[(z-z_2)(z-\bar{z}_2)]^{-m_2},$$

which can be chosen a rational function if and only if the ratio $m_2:m_1$ is rational.

Theorem 3 is quite unusual in the sense that the proof (as we have given it) of a theorem concerning polynomials depends on a proposition (Theorem 2) for more general rational functions followed by a linear transformation of the complex variable. Naturally the proof of Theorem 3 can be otherwise phrased. Let $r(z)$ be a real rational function having its only pole in the extended plane at the finite point a, $a<0$, and let all real parts of the zeros of $r(z)$ be nonnega-

tive, the real parts of all nonreal zeros positive. At a point z_0 in the upper half-plane exterior to the circles $\Gamma(z_k)$ of Theorem 2, the total force has a nonzero component perpendicular to the line Oz_0 in the counterclockwise sense with respect to O. Let now a linear transformation carry the entire configuration into the situation of Theorem 3. Although the magnitude of the force is not invariant under such a linear transformation, both direction and sense of the force are invariant ([2], Sec. 4.1.2); the total mass of all particles in the extended plane is to be zero. Consequently, under the conditions of Theorem 3, at any point z_0 in the upper half-plane exterior to the circles Γ_k, the force has a nonzero component orthogonal to the arc $\alpha z_0 \beta$ in the sense away from the axis of reals; Theorem 3 can obviously be proved by this remark.

In Theorem 3 the points α and β are not uniquely determined; however, it is not possible to consider the lens-shaped region Λ_k interior to all (variable) Γ_k as k remains fixed while α and β vary, and to substitute Λ_k for Γ_k in the theorem. As a counterexample we mention $p(z) \equiv (z-1-2i)(z-1+2i)(z+1-2i)(z+1+2i) \equiv z^4+6z^2+25$, the nonreal zeros $\pm i\sqrt{3}$ of whose derivative lie at the intersections of the two Jensen circles for $p(z)$ yet do not lie in either Λ_k.

We add a number of remarks complementary to Theorems 2 and 3. By the lemma we have the

COROLLARY. *Let the hypothesis of Theorem 2 be enlarged so as to admit nonreal poles of $r(z)$ also in the half-plane $x \geqq 0$. Then a nonreal point in $x>0$, interior to all circles $\Gamma(z_k)$ defined as in Theorem 2 for the poles of $r(z)$ in $x>0$ and exterior to all circles $\Gamma(z_k)$ for the zeros of $r(z)$, cannot be a zero of $r'(z)$.*

Under the conditions of this corollary, it may occur that no poles of $r(z)$ lie in the closed half-plane $x \leqq 0$; under such conditions any real point $\gamma(<0)$ can replace 0 in defining the circles $\Gamma(z_k)$, and we may even choose $\gamma = -\infty$ (compare [2], Sec. 5.1.2, Theorem 3; [1], corollary).

In this corollary the roles of zeros and poles may be interchanged, but it is to be noted that a multiple zero of $r(z)$ whether real or nonreal is also a zero of $r'(z)$.

Theorem 2 can be generalized by transforming a given configuration containing a "circle" C by a linear transformation of the complex variable that carries C into the axis of reals. We deal with the extended plane, and allow the term "circle" to include straight line, and we use the term *circular region* to denote either the closed interior of a circle, the closed exterior of a circle, or a closed half-plane. Let $r(z)$ be a rational function whose zeros and whose poles not on a "circle" C both occur in pairs of points mutually inverse in C, where also the zeros of $r(z)$ not on a second "circle" C_1 orthogonal to C are separated by C_1 from the poles of $r(z)$ not on C_1, and where no zeros nor poles of $r(z)$ lie on C_1 except perhaps in the intersections α and β of C and C_1. Let a "circle" $\Gamma(z_k)$ be drawn (necessarily orthogonal to C) through each pair (z_k, z_k') of zeros and of poles of $r(z)$ mutually inverse in C, tangent at z_k and z_k' to the "circles" $\alpha z_k \beta$ and $\alpha z_k' \beta$, and let $\Gamma(z_k)$ denote also that circular region bounded by the

"circle" $\Gamma(z_k)$ which contains no point of C_1. Then all finite zeros of $r'(z)$ not on C lie in the regions $\Gamma(z_k)$.

The concept of *infrapolynomial* was first introduced by Fekete and von Neumann [3], as a generalization of classical extremal polynomials $p(z) \equiv z^n + a_1 z^{n-1} + \cdots + a_n$ of least norm on a given point set E. The zeros of such infrapolynomials are positions of equilibrium in a suitably chosen field of force analogous to the field of Gauss, where the particles lie on E but need no longer be of integral mass. Thus Fekete and von Neumann showed (*loc. cit.*) that the analogue of Jensen's theorem holds if $p(z)$ is real and if E is symmetric in the axis of reals. It is likewise true that under such conditions the analogue of Theorem 1 is valid ([4], Theorem 7; [5]); similar considerations show that also the analogue of Theorem 3 is valid.

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CORRECTION

In the final printing of *The rolling of one curve or surface upon another* by William Clifford and James J. McMahon (this MONTHLY, vol. 68, 1961, pp. 338–341), a large number of dots over letters in displayed formulas somehow failed to appear.

The correct formulas are, on page 338:

$$0 = \dot{X} = \dot{A}X_0 + \dot{C} \tag{2}$$

$$\dot{A}X_0 = -\dot{C} \tag{2a}$$

$$X_0 = -(\dot{A})^{-1}\dot{C} \tag{3}$$

$$0 = \dot{A}^T A + A^T \dot{A} = (A^T \dot{A})^T + (A^T \dot{A}),$$

On page 339:

$$X = -A(\dot{A})^{-1}\dot{C} + C \tag{4}$$

$$(\dot{A}^T)Y = 0 \Rightarrow (\dot{C}^T)Y = 0. \tag{5}$$

$$X + \dot{X}\delta t = AX_0 + C + (\dot{A}X_0 + \dot{C})\delta t. \tag{6}$$

(7)
$$X + \dot{X}\delta t = C + [(I + \dot{A}A^T\delta t)(X - C)] + \dot{C}\delta t.$$
$$(I + \dot{A}A^T\delta t)Y = Y,$$

On page 340:

(10a)
$$P_i^T\left\{\dot{Q}_0 + \sum_{j=1}^{s}(\dot{\lambda}_j P_j + \lambda_j \dot{P}_j)\right\} = 0, \qquad (i = 1, \cdots, s)$$

(10b)
$$\dot{P}_i^T\left\{\dot{Q}_0 + \sum_{j=1}^{s}(\dot{\lambda}_j P_j + \lambda_j \dot{P}_j)\right\} = 0,$$

The Editor regrets these errors.

MATHEMATICAL NOTES

EDITED BY ROY DUBISCH, University of Washington

Material for this department should be sent to M. H. Protter, Department of Mathematics, University of California, Berkeley 4, California

LEAST COMMON MULTIPLES AND HIGHEST COMMON FACTORS

MARLOW SHOLANDER, Western Reserve University

It is well known that $ab = (a, b)[a, b]$, where we use standard notation for highest common factor (HCF) and least common multiple (LCM) of positive integers a and b. It is less well known that*

(1)
$$abc = (a, b, c)[(a, b), (b, c), (c, a)][a, b, c],$$

(2)
$$abc = (a, b, c)[ab, bc, ca],$$

(3)
$$abc = (ab, bc, ca)[a, b, c].$$

The proof of (3) is typical. Let the highest power of prime p dividing a, b, and c be p^α, p^β, and p^γ. Then both sides of (3) contain p to the exponent $\alpha+\beta+\gamma = \min(\alpha+\beta, \beta+\gamma, \gamma+\alpha)+\max(\alpha, \beta, \gamma)$.

The generalizations are clear. Consider a set $S\{a_1, a_2, \cdots, a_n\}$. Form the set T of $\binom{n}{k}$ LCM's of the subsets of S which have k elements. Let P_k be the HCF of T. Alternately, P_k is the LCM of the set of $\binom{n}{k}$ HCF's of subsets of S which have $n-k$ elements. Then $\prod_{k=1}^{n} a_k = \prod_{k=1}^{n} P_k$.

Let U be the set of $\binom{n}{i}$ products obtained from subsets of S having i elements. Let Q_i be the HCF of U and R_i be the LCM of U. Then $Q_i = \prod_{k=1}^{i} P_k$ and $R_{n-i} = \prod_{k=i+1}^{n} P_k$. Hence $\prod_{k=1}^{n} a_k = Q_i R_{n-i}$ for $i = 0, 1, \cdots, n$.

* We note that $[(a, b), (b, c), (c, a)] = ([a, b], [b, c], [c, a])$ is an example of a "median." Cf. M. Sholander, Medians and betweenness, Proc. Amer. Math. Soc., vol. 5, 1954, pp. 801–807.

ON A THEOREM OF HOBSON

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An interesting and useful theorem ([2], p. 363; [1], p. 246) requires the existence of the derivative. However, since the derivative may fail to exist, it seems desirable to have expressions which may serve us when there is no derivative. The purpose of this note is to extend the theorem to semi-continuous functions.

DEFINITION. *If the function f is defined for $a \leqq x < a+h$, a number r is a right derivate of f at a if there exists a sequence $b_1, b_2, \cdots$ of positive numbers tending to 0 and less than h such that*

$$\lim_{n\to\infty} \frac{f(a+b_n)-f(a)}{b_n} = r.$$

Left derivates are analogously defined.

THEOREM. *Let the function f be lower semi-continuous on the open interval $a<x<a+h$ and let the right limit $f(a+)$ fail to exist. Then for each real number k there exists a number ξ with $a<\xi<a+h$ such that:*

(i) *if $D(+)f(\xi)$ is any right derivate of f at ξ and $D(-)f(\xi)$ is any left derivate of f at ξ, then $D(-)f(\xi) \leqq k \leqq D(+)f(\xi)$;*

(ii) *if $D(+)f(\xi)$ is any finite right derivate of f at ξ and $D(-)f(\xi)$ is any finite left derivate of f at ξ, there exist nonnegative numbers p, q with $p+q=1$ such that $pD(+)f(\xi)+qD(-)f(\xi)=k$.*

A dual theorem holds if f is upper semi-continuous for $a<x<a+h$.

Proof. Let $F(x)=f(x)-kx$. Since $f(a+)$ does not exist neither does $F(a+)$, and there exists a number b such that

$$\liminf_{x\to a+} F(x) < b < \limsup_{x\to a+} F(x).$$

We choose successively three numbers b_1, c_1, a_1 such that:

$$a < b_1 < a+h \quad \text{and} \quad F(b_1) > b,$$
$$a < c_1 < b_1 \quad \text{and} \quad F(c_1) < b,$$
$$a < a_1 < c_1 \quad \text{and} \quad F(a_1) > b;$$

this is possible by the definition of b. On the closed interval $a_1 \leqq x \leqq b_1$, the function F is lower semi-continuous, so it attains its minimum at some point ξ in that interval ([3], p. 76). But ξ is interior to that interval, for $F(\xi) \leqq F(c_1) < b$ while $F(a_1) > b$ and $F(b_1) > b$. By a proof to be found in many good calculus books, all right derivates of F at ξ are $\geqq 0$ and all left derivates of F at ξ are $\leqq 0$. Since $D(+)F(\xi) = D(+)f(\xi) - k$ (the $D(+)$ being defined by the same sequence

* The author is indebted to the referee for his helpful suggestions.

for f as for F), conclusion (i) holds. Conclusion (ii) follows from (i), since the inequality in (i) implies for finite $D(+)f(\xi)$ and $D(-)f(\xi)$ that k is a weighted mean of the derivates.

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TWO METHODS FOR THE EVALUATION OF $\sum_{k=0}^{\infty} k^n x^k$

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Stalley [1] and Klamkin [2] have given different proofs for summing a generalized geometric series, $K_n(x)$, and obtained the following result:

$$K_n(x) = \sum_{k=1}^{\infty} k^n x^k = (1-x)^{-(n+1)} \sum_{r=1}^{n} \left[\sum_{m=1}^{r} (-1)^{m+1} \binom{n+1}{m-1} (r-m+1)^n \right] x^r$$

for $|x| < 1$ and n a nonnegative integer. In this note we introduce two additional ways for obtaining $K_n(x)$. The first method uses generating functions, and the second method requires the following identity:

$$\frac{d^{m-1}}{du^{m-1}} \left(\frac{1}{e^u - 1} \right) = (-1)^{m-1} \sum_{r=1}^{m} (e^u - 1)^{-r} \sum_{s=0}^{r-1} (-1)^s \binom{r-1}{s} (r-s)^{m-1}, \tag{1}$$

$$m \geqq 1.$$

For a proof of (1), see the elementary problem E1401 [1960, 803] of this MONTHLY, vol. 67.

(i) Let u_k satisfy a linear difference equation of order $(n+1)$ with real, constant coefficients:

$$u_{n+1+k} + a_1 u_{n+k} + a_2 u_{n+k-1} + \cdots + a_{n+1} u_k = 0, \qquad a_{n+1} \neq 0.$$

Then the generating function of u_k (see [3, p. 27]) is given by

$$\sum_{k=0}^{\infty} u_k x^k = \sum_{k=0}^{n} \left(\sum_{j=0}^{n-k} a_j u_{n-k-j} \right) x^{n-k} \Big/ \sum_{i=0}^{n+1} a_i x^i, \tag{2}$$

where $a_0 \equiv 1$. From the well-known identity,

$$\sum_{j=0}^{n+1} (-1)^j \binom{n+1}{j} j^r = 0, \qquad 0 \leqq r < n+1,$$

it follows that $u_k = k^n$ satisfies

$$\sum_{j=0}^{n+1} (-1)^j \binom{n+1}{j} u_{n+1+k-j} = 0.$$

Thus, with

$$a_j = (-1)^j\binom{n+1}{j}, \qquad j = 0, 1, \cdots, (n+1), \quad |x| < 1,$$

we obtain from (2)

$$(3) \qquad (1-x)^{n+1}\sum_{k=0}^{\infty} k^n x^k = \sum_{k=0}^{n}\left[\sum_{j=0}^{n-k}(-1)^j\binom{n+1}{j}(n-k-j)^n\right]x^{n-k}, \qquad n \geqq 0.$$

If we define

$$A_{ns} = \sum_{j=0}^{s}(-1)^j\binom{n+1}{j}(s-j)^n, \qquad 0 \leqq s \leqq n,$$

then $A_{n0}=0$ if $n\neq 0$, $A_{nn}=1$, $n=0, 1, \cdots$, and (3) becomes

$$(4) \qquad (1-x)^{n+1}\sum_{k=0}^{\infty} k^n x^k = \sum_{k=0}^{n} A_{n,n-k}x^{n-k}, \qquad n \geqq 0.$$

A_{ns} are known as Eulerian numbers (see [4], [5]). If we define $S_k(n)=1^n+2^n+\cdots+k^n$, $k\geqq 1$, $n\geqq 1$, then

$$(5) \qquad \sum_{k=1}^{\infty} S_k(n)x^k = \sum_{k=0}^{\infty} k^n x^k/(1-x) = (1-x)^{-(n+2)}\sum_{k=0}^{n} A_{n,n-k}x^{n-k}, \quad n \geqq 1.$$

(ii) Let $e^r x<1$, where r is real and $0<x<1$. Then

$$\sum_{k=0}^{\infty} e^{kr}x^k = \sum_{k=0}^{\infty}(e^r x)^k = (1-e^r x)^{-1},$$

and

$$\sum_{k=0}^{\infty} k^n x^k = \left[\frac{d^n}{dr^n}\left(\frac{1}{1-e^r x}\right)\right]_{r=0}, \qquad 0 \leqq x < 1.$$

Writing $(1-e^r x)^{-1}$ as $(1-e^{r+\log x})^{-1}$, and recalling that $[t/(e^t-1)]=\sum_{k=0}^{\infty} B_k t^k/k!$, where B_k are Bernoulli numbers, we have $(e^{r+\log x}-1)^{-1}=\sum_{k=0}^{\infty} B_k(r+\log x)^{k-1}/k!$. Thus,

$$(6) \qquad \sum_{k=0}^{\infty} k^n x^k = -\sum_{k=0}^{\infty}\frac{B_k(k-1)\cdots(k-n)(\log x)^{k-n-1}}{k!}$$

for $e^{-2\pi}<x<1$. This is so since $e^z-1=0$ if $z=2m\pi i$, $m=0, \pm 1, \cdots$, and we require $-2\pi<\log x<0$. Let $u=\log x$. Since $1/(e^u-1)=\sum_{k=0}^{\infty} B_k u^{k-1}/k!$, we observe that

$$\frac{d^n}{du^n}\left(\frac{1}{e^u-1}\right) = \sum_{k=0}^{\infty}\frac{B_k(k-1)\cdots(k-n)u^{k-n-1}}{k!}.$$

Thus, from (1), with $m=n+1$, (6) becomes

$$(7)\qquad \sum_{k=0}^{\infty} k^n x^k = (-1)^{n+1} \sum_{r=1}^{n+1} \frac{1}{(x-1)^r} \sum_{s=0}^{r-1} (-1)^s \binom{r-1}{s} (r-s)^n, \qquad n \geqq 0.$$

By the principle of analytic continuation, (7) persists for $|x|<1$. We note that (7) gives the result in a form different than (3). (7) may be written as follows:

$$(8)\qquad \sum_{k=0}^{\infty} k^n x^k = (1-x)^{-n-1} \sum_{j=0}^{n} (x-1)^{n-j} \sum_{s=0}^{j} (-1)^s \binom{j}{s} (j-s+1)^n$$

$$= (1-x)^{-n-1} \sum_{j=0}^{n} (x-1)^{n-j} \Delta^j 1^n,$$

where

$$\Delta^j 1^n = \sum_{s=0}^{j} (-1)^s \binom{j}{s} (j-s+1)^n.$$

We wish to show that (8) can be transformed into (4). Expanding $(x-1)^{n-j}$ by the binomial theorem, we obtain

$$\sum_{j=0}^{n} (x-1)^{n-j} \Delta^j 1^n = \sum_{j=0}^{n} \Delta^j 1^n \sum_{r=0}^{n-j} \binom{n-j}{r} (-1)^{n-j-r} x^r$$

$$= \sum_{r=0}^{n} \left[\sum_{j=0}^{n-r} (-1)^{n-j-r} \binom{n-j}{r} \Delta^j 1^n \right] x^r = \sum_{k=0}^{n} A_{n,n-k} x^{n-k},$$

since

$$(9)\qquad A_{n,n-k} = \sum_{j=0}^{k} (-1)^{k-j} \binom{n-j}{n-k} \Delta^j 1^n.$$

The proof of (9) is given in [2], page 92 where our (9) occurs as (8) in the note. This is readily seen by setting $t-1=k$ in (8) and by noting that $A_{n,n-k}=A_{n,k+1}$ (see [4, p. 250, (2.13)]).

Remarks. We note that

$$(1-x)^{n+1} K_n(x) = \sum_{k=0}^{n-1} A_{n,n-k} x^{n-k}, \qquad n \geqq 1.$$

(10) and (11) cited below are due to Euler (see [4, p. 247]):

$$(10)\qquad \frac{1-x}{e^u - x} = \sum_{n=0}^{\infty} H_n(x) u^n/n!, \qquad x \neq 1,$$

$$(11)\qquad R_n(x) = (x-1)^n H_n(x), \qquad xR_n(x) = \sum_{s=1}^{n} A_{n,s} x^s.$$

Thus,

$$K_n(x) = (-1)^n x H_n(x)/(1-x), \qquad n \geqq 0, \tag{12}$$

$$\sum_{k=0}^{\infty} k^n x^k = (-1)^n x H_n(x)/(1-x), \qquad n \geqq 1, \tag{13}$$

where

$$H_n(x) = \sum_{r=0}^{n} (x-1)^r \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} s^n, \qquad H_0(x) = 1. \tag{14}$$

(14) is given in [4, p. 250, (2.10)]. For $n \geqq 1$, (13) and (12) are, of course, identical.

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2. M. S. Klamkin, On a generalization of the geometric series, this MONTHLY, vol. 64, 1957, pp. 91–92.
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A COUNTING THEOREM

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In a previous paper* the author presented the following counting theorem:

Let j, s, and t be integers for which $0 \leqq s \leqq t$. Then, if the elements of j sets, each containing s elements, are distributed among t classes in such a way that no two elements of the same set fall in the same class, then for each integer r satisfying $0 \leqq r \leqq s$, there exist at least r of the classes each of which contains at least

$$m_r = \left[\frac{(j+1)(s-r+1) + t - s - 1}{t-r+1} \right]$$

elements.† Moreover, there is no integer $m_r^1 > m_r$ for which the same conclusion can be drawn.

The restriction that the sets all have one and the same number of elements is somewhat undesirable. It is the purpose of the present note to show that this restriction may be removed without greatly complicating the result.

First note that m_r may be rewritten in the form

* Paul M. Pepper and Bernard J. Topel, Imbedding Theorems under Weakened Hypotheses: Part I. Reports of a Mathematical Colloquium, Second Series, Issue 4, pp. 52–53 (University of Notre Dame).

† The expression $[a]$ means the greatest integer less than or equal to a.

$$m_r = \left[\frac{j(s-r+1)+(t-r)}{t-r+1}\right],$$

and that $s-r+1$ is the number of elements in excess of $r-1$ which each set contains; $t-r+1$ is the excess over $r-1$ in the number of classes; $j(s-r+1)$ is the total number of elements all j of the sets contain in excess of $r-1$ each. The ratio $j(s-r+1)/(t-r+1)$ is the average number of elements in each of $t-r+1$ excess classes if the excess elements are distributed into those classes, and, if these excess elements are distributed as evenly as possible, the expression for m_r is the maximum number of these excess elements which one of the $t-r+1$ classes may have. Thus, as few as possible of the elements are distributed into the $t-r+1$ classes, in order that $r-1$ but not any set of r classes be filled deeply.

To generalize the theorem, one should try to force some $t-r+1$ classes to have as few elements as possible, stacking the other elements into the remaining classes as deeply as possible. This line of heuristic reasoning leads to the following formulation and proof of a theorem completely analogous to the earlier one, but having none of the restrictions on its range of applicability.

THEOREM. *Let $S_1, \cdots, S_j$ be j finite sets of elements with respective numbers of elements equal to $s_1, \cdots, s_j$ and such that $s_1 \leqq \cdots \leqq s_j$. Let t be any integer satisfying $s_j \leqq t$, and let r be any integer in $0 < r \leqq t$. Let the elements of the sets S_i be distributed into t classes in such a way that no two elements of the same set fall into the same class. Let i_0 be the first value of i(if it exists) for which $s_i \geqq r$, so that $s_{i_0-1} < r \leqq s_{i_0}$. Then there exist at least r of the t classes each containing at least*

$$m_r = \left[\frac{\sum_{i=i_0}^{j}(s_i-r+1)+(t-r)}{t-r+1}\right] \tag{1}$$

elements.

If r is greater than all of the s_i, then, on the one hand no i_0 exists, and, on the other hand, the maximum integer for which the conclusion is valid is $m_r = 0$, which can be obtained by dropping the sum in (1) *to get*

$$m_r = \left[\frac{t-r}{t-r+1}\right] = 0.$$

Moreover, in the general case, if m_r^1 is any integer greater than m_r, there exist distributions such that among each r classes there exists at least one which contains at most $m_r^1 - 1$ elements.

To make a proof of this theorem, we note that to negate the theorem, there must exist at least one distribution such that of every set of $t-r+1$ classes each contains at most $m_r - 1$ elements.

Let some $t-r+1$ classes be selected and distribute the elements of the sets $S_{i_0}, S_{i_0+1}, \cdots, S_j$ into the t classes in such a way as to minimize the number

which fall into the selected $t-r+1$ classes. This is done by distributing some $r-1$ elements from S_i into the remaining classes and the other s_i-r+1 elements into the $t-r+1$ classes. Then the number of elements in these classes is $\sum_{i=i_0}^{j}(s_i-r+1)$ so that the average number of elements in each class is

$$\mu = \frac{\sum_{i=i_0}^{j}(s_i - r + 1)}{t - r + 1}. \tag{2}$$

At least one class has at least m_r elements in it if m_r is the least integer $\geqq\mu$. But the least integer $\geqq\mu$ can be expressed in the form (1).

Since $s_i \leqq t$, it follows from (2) that $j-i_0+1 \geqq \mu$ and since j and i_0 are integers, $j-i_0+1 \geqq m_r$. This completes the proof that there are never $t-r+1$ classes with less than m_r elements in each. Thus among each r classes, each has at least m_r elements.

To show that this cannot be said of any integer $m_r^1 > m_r$, it suffices to show a distribution—this time of the elements of all j of the sets—for which some $t-r+1$ classes each contains at most $m_r(<m_r^1)$ elements.

To exhibit such a distribution separate the t classes into two categories; let the first category contain $r-1$ classes and the second $t-r+1$ classes. Since $0 < s_1 \leqq \cdots \leqq s_{i_0-1} \leqq r-1$ it is possible to distribute the elements of $S_1, \cdots, S_{i_0-1}$ into only the $r-1$ classes of the first category. Moreover, as before, it is possible to put $r-1$ of the elements of each of the sets $S_{i_0}, \cdots, S_j$ into the classes of the first category. Order the classes of the second category and following that order starting with i_0, distribute the remaining $s_{i_0}-r+1$ points (of S_{i_0}) into the first $s_{i_0}-r+1$ classes of the second category; continue with the remaining points of $S_{i_0+1}, \cdots$, etc., until each class of the second category has exactly one element in it. Next return to the first class of the second category to continue the distribution following the same order as previously until each class of the second category has exactly 2 elements in it (provided there are enough elements). At the end of the distribution of those $\sum_{i=i_0}^{j}(s_i-r+1)$ elements of the sets $S_{i_0}, \cdots, S_j$ which were not allocated to the classes of the first category, each class of the second category will contain either

$$\left[\frac{\sum_{i=i_0}^{j}(s_i - r + 1)}{t - r + 1}\right] \tag{3}$$

elements or one more than this number. If the number in brackets is an integer all of the classes of the second category will have the same number of elements, otherwise some will have the number of elements described by (3) and others will have one more than this number. In no event will any class have more than the number m_r of (1), so that there is a distribution in which $t-r+1$ classes have less than m_r^1 elements whatever integer $m_r^1 > m_r$ may be chosen.

CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

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ANOTHER VERY INDEPENDENT AXIOM SYSTEM

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In a recent Classroom Note [1], Harary presented a very independent (in fact, an absolutely independent) axiom system and conjectured that some, but not many, others might exist. This note outlines another such system, which selects the subgroups among the subsets of a group. The notation and terminology are those of [1].

The primitives of our system are a group G, with identity element denoted by 1, and a subset S of G. The axioms are:

i. $1 \in S$

v. If $x \in S - \{1\}$, then $x^{-1} \in S$

c. If $x, y \in S - \{1\}$ and $x \neq y^{-1}$, then $xy \in S$.

The slight alteration of axioms v and c from their usual forms was necessary to avoid implying i; thus they are similar to the distinctly transitive axiom used by Harary.

We now exhibit eight subsets of the group G of positive rational numbers, satisfying the eight possible combinations of i, $\bar{\text{i}}$, v, $\bar{\text{v}}$, c, $\bar{\text{c}}$. Thus our system, too, is absolutely independent.

Conditions	S
ivc	G
$\bar{\text{i}}$vc	$G - \{1\}$
i$\bar{\text{v}}$c	$\{r \mid r \in G, r \geq 1\}$
iv$\bar{\text{c}}$	$\{1\} \cup \{p \mid p \text{ a prime}\} \cup \{1/p \mid p \text{ a prime}\}$
i$\bar{\text{v}}\bar{\text{c}}$	$\{1\} \times \{p \mid p \text{ a prime}\}$
$\bar{\text{i}}$v$\bar{\text{c}}$	$\{p \mid p \text{ a prime}\} \cup \{1/p \mid p \text{ a prime}\}$
$\bar{\text{i}}\bar{\text{v}}$c	$\{r \mid r \in G, r > 1\}$
$\bar{\text{i}}\bar{\text{v}}\bar{\text{c}}$	$\{2, 3\}$.

It may be argued that our axioms i, v, c are closely related to the axioms r, s, t respectively for the relation of (left or right) congruence modulo S. Even so, the models above differ from those in [1] in the sense that all relations used are of the same type.

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1. Frank Harary, A very independent axiom system, this MONTHLY, vol. 68, 1961, pp. 159–164.

A RECURSION FORMULA FOR A CERTAIN DEFINITE INTEGRAL

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It is known that

$$y(\alpha) \equiv \int_0^1 \frac{t^{2\alpha-1}}{\sqrt{(1-t^2)}}\,dt = \tfrac{1}{2}\sqrt{\pi}\,\frac{\Gamma(\alpha)}{\Gamma(\alpha+\tfrac{1}{2})} \qquad (\alpha > 0).$$

Differentiating this logarithmically with respect to α we get

$$\frac{y'(\alpha)}{y(\alpha)} = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} - \frac{\Gamma'(\alpha+\tfrac{1}{2})}{\Gamma(\alpha+\tfrac{1}{2})} = G(\alpha),$$

say. Next, taking the nth derivative of both sides of the above in the form $y' = yG$, we obtain, by Leibnitz's rule,

$$y^{(n+1)} = y^{(n)}G + \binom{n}{1} y^{(n-1)}G^{(1)} + \cdots + yG^{(n)}, \tag{1}$$

where $y^{(r)}$ and $G^{(r)}$ $(r \geqq 1)$ are given by

$$y^{(r)} = 2^r \int_0^1 \frac{t^{2\alpha-1}}{\sqrt{(1-t^2)}} \log^r t\,dt, \tag{2}$$

$$G^{(r)} = \frac{d^r}{d\alpha^r}\left\{\frac{\Gamma'(\alpha)}{\Gamma(\alpha)} - \frac{\Gamma'(\alpha+\tfrac{1}{2})}{\Gamma(\alpha+\tfrac{1}{2})}\right\} \tag{3}$$

$$= (-1)^{r+1}r!\left\{\sum_{m=0}^{\infty} \frac{1}{(\alpha+m)^{r+1}} - \sum_{m=0}^{\infty} \frac{1}{(\alpha+\tfrac{1}{2}+m)^{r+1}}\right\}. \tag{4}$$

Equation (4) follows from (3) by differentiation of the well-known formula*

$$\frac{\Gamma'(x)}{\Gamma(x)} - \frac{\Gamma'(y)}{\Gamma(y)} = \sum_{m=0}^{\infty}\left\{\frac{1}{y+m} - \frac{1}{x+m}\right\}.$$

Using (2), (3), (4) in (1), setting $\alpha = \frac{1}{2}$ and noting that

$$\frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} - \frac{\Gamma'(1)}{\Gamma(1)} = -2\log 2,$$

we get the desired recursion formula

$$\int_0^1 \frac{\log^{n+1} t}{\sqrt{(1-t^2)}}\,dt = -\log 2 \int_0^1 \frac{\log^n t}{\sqrt{(1-t^2)}}\,dt + \sum_{r=1}^{n} \frac{(-1)^{r+1}}{2^{r+1}}\binom{n}{r} r!\{\zeta(r+1,\tfrac{1}{2}) - \zeta(r+1)\}\int_0^1 \frac{\log^{n-r} t}{\sqrt{(1-t^2)}}\,dt,$$

* See, e.g., Earl D. Rainville, Special Functions, New York, 1960.

where $\zeta(r, s) = \sum_{m=0}^{\infty} (m+s)^{-r}$, $\zeta(r, 1) = \zeta(r)$.

For $n = 0$ we get the well-known result

$$\int_0^1 \frac{\log t}{\sqrt{(1-t^2)}}\, dt = -\tfrac{1}{2}\pi \log 2.$$

For $n = 1, 2$, we get

$$\int_0^1 \frac{\log^2 t}{\sqrt{(1-t^2)}}\, dt = \tfrac{1}{2}\pi \log^2 2 + \tfrac{1}{4}\{\zeta(2, \tfrac{1}{2}) - \zeta(2)\} \cdot \tfrac{1}{2}\pi$$

$$= \tfrac{1}{2}\pi \log^2 2 + \tfrac{1}{4}(\tfrac{1}{2}\pi^2 - \tfrac{1}{6}\pi^2) \cdot \tfrac{1}{2}\pi = \tfrac{1}{2}\pi \log^2 2 + \tfrac{1}{24}\pi^3,$$

$$\int_0^1 \frac{\log^3 t}{\sqrt{(1-t^2)}}\, dt = -\tfrac{1}{8}\pi[\pi^2 \log 2 + 4 \log^3 2 + \zeta(3, \tfrac{1}{2}) - \zeta(3)],$$

respectively. There are similar results for $n \geqq 3$.

GRAPHICAL MULTIPLICATION OF FUNCTIONS

HARRY I. MILLER, Illinois Institute of Technology

Consider a plane in which a horizontal straight line O is given.* On it, a linear unit is laid off from 0 to 1. Consider the graphs f and g of two functions. We wish to construct points of the curve $f \cdot g$, which is the graph of the product of those two functions. If X is the point on O that is x units from 0, then the

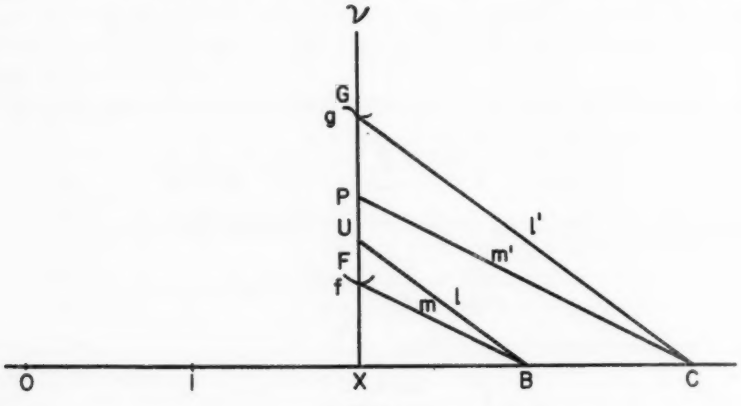

FIG. 1

* In what follows we adopt Menger's typographical convention ([1], p. 10): All designations of lines, curves, and functions are lower case italics, while all references to numbers are lower case roman. A function and its graph will be denoted by the same letter. The line O is the graph of the constant function whose value is 0. Points are denoted by capitals in roman type. The value of the function f for the number x will be denoted by fx.

altitudes above X of the points F on f, and G on g, are fx and gx units, respectively. We have to construct the point P whose altitude above X is $fx \cdot gx$ linear units. (In order to obtain many points of $f \cdot g$, this construction has to be performed for many points X, X_1, X_2, $\cdots$). In Figure 1 we illustrate three constructions of the point P that fulfill the requirement.

1. The projective construction. On the vertical line v through X, lay off a linear unit to the point U. Choose any point B ($\neq$X) on O. Join B to U by the straight line l, and to F by m. Then, through G, draw the line l', parallel to l. Through C (the intersection of l', with O), draw the line m' parallel to m. The point where m' intersects v can, by similar triangles, easily be shown to be $fx \cdot gx$ units above X.

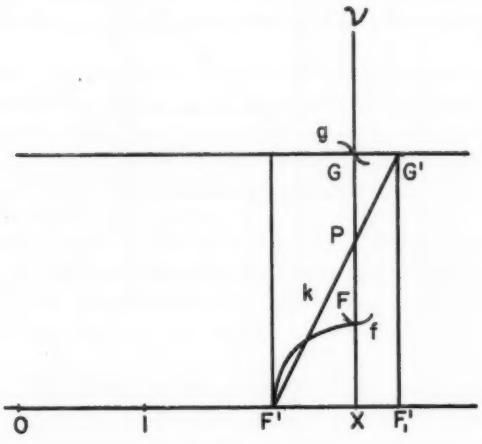

FIG. 2

2. The compass construction. (Fig. 2). Draw the quarter circle about X from F to F′ (to the left of X). The area in square units of the rectangle with the sides F′X and XG is $fx \cdot gx$. Since we have to construct a vertical segment of just that number of linear units, we follow the graphical construction of areas ([1], p. 8); that is to say, we lay off a linear unit on O from F′ to F_1', and join F′ with the point G′ (gx linear units above F_1') by the line k. The point P where k intersects v is, as can be shown by similar triangles, $fx \cdot gx$ units above X.

3. The j-method. (Fig. 3). Consider the graph of the identity function ([1], p. 4, 75), that is, the line j which, for any number c, has an altitude of c units above the point that is c units from 0. From F draw the horizontal line to F* where it intersects j. Draw the vertical line v^* through F* to F_0^*, its intersection with O. Since F* lies on j, the point F_0^* has the same distance from 0 as from F*, that is fx units. The area in square units of the rectangle with the

sides $0F_0^*$ and $F_0^*G^*$ (where G^* is gx units above F_0^*) is $fx \cdot gx$. We construct the area of this rectangle just as, in the course of the compass construction, we constructed the area of the rectangle with the sides F'X and XG. (Clearly, the two rectangles are congruent by a horizontal translation.) We join the point 0 with G_1^* (gx units above the point 1) by the line k^*. The point P^* where k^* intersects v^* has an altitude of $fx \cdot gx$ units above O. Hence the horizontal line through P^* intersects v in a point P which is the same number of units above X.

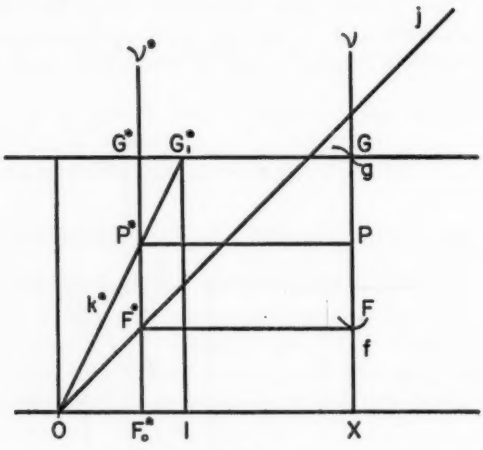

FIG. 3

Remark 1. Even if vertical and horizontal distances are measured in unlike units and, therefore, the line j has an inclination $\neq 45°$, the j-method works; that is to say, the number of vertical units in the segment from X to P is equal to $fx \cdot gx$.

Remark 2. The j-method links graphical multiplication to Menger's graphical substitution ([1],† p. 89), which yields the graph of the composite function of f and g which assumes the value $f(g(x))$ for any x. Menger's construction, based on j, is carried out entirely in the plane, even though it has been claimed ([3], p. 79) that geometric substitution has to resort to three-dimensional space. A remarkable example ([2], p. 460) is the graphical substitution of the parabola $-j^2$ (having the altitude $-x^2$ above x) into the exponential curve (of altitude e^x above x), which results in a probability curve (having the altitude e^{-x^2} above x).

References

1. K. Menger, Calculus. A Modern Approach, Boston, 1955, p. 10.

† The mimeo-editions of this book (Chicago, 1952, p. 224 and 1953, p. 273) include also a construction of the substitution of an ordered pair of surfaces into a surface.

2. ———, Axiomatic theory of functions and fluents. The Axiomatic Method (ed. Henkin *et al.*), Amsterdam, 1959.
3. A. Tarski, On the calculus of relations, J. Symb. Logic, vol. 6, 1941, pp. 73–89.

FOUR-POINT FORMULAS FOR MACHINE COMPUTATION

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The writer has developed and used the following generalization of Simpson's rule for integration on an IBM 650 computer. The programming of the formula was simple and allowed many variations as a self-contained subroutine. As an example, a combined differentiation, integration and interpolation subroutine was easily devised, using the same entrances with a code call word and the same formula pattern.

The formula also lends itself to an entrance check for variable interval, *i.e.* if $h_1=h_2=h_3$ no coefficient computation is necessary. The running time for the formula on a computer is about the same as for Simpson's rule, except when a change of interval is encountered. When the interval changes, the coefficients have to be recomputed and hence the running time is increased. Storage space required is approximately two bands.

Derivation of the formula follows standard methods. Assume that we have four points u_a, u_b, u_c, and u_d. If we pass a cubic $u_x=A(x-b)(x-c)(x-d)+\cdots+D(x-a)(x-b)(x-c)$ through the four points and solve for the coefficients A, B, C, and D we get

$$(*) \qquad \int_0^h u_x dx = \tfrac{1}{4}Ah^4 + \tfrac{1}{3}Bh^3 + \tfrac{1}{2}Ch^2 + Dh,$$

where $h=h_1+h_2+h_3$ and

$$A = \frac{u_a}{h_1(h_1+h_2)h}, \quad B = \frac{-u_b}{h_1h_2(h_2+h_3)}, \quad C = \frac{u_c}{(h_1+h_2)h_2h_3}, \quad D = \frac{-u_d}{h(h_2+h_3)h_3},$$

after the substitutions $a=0$, $b-a=h_1$, $c-a=h_1+h_2$, $d-a=h$ are made. Starting and stopping formulas follow in a similar manner. Comparable predictor-corrector formulas were derived and proved worthwhile for machine computation.

If $R(x)$ is the error, then, again writing $h=h_1+h_2+h_3$,

$$R(x) = \int_0^h (x^4/4!)dx - (*) = (h^5/5!) - (*).$$

The actual use of the error formula involves no great increase in computation since the error formula has only one extra term and computation of this term is rapid on a computer. The extra term, of course, need not be recomputed each time if no change has occurred in interval size.

A METRIC PARADOX

ALBERT WILANSKY, Lehigh University

Let z be a fixed point in a metric space (X, d) and define $f(x) = d(x, z)$. Now a real function g on a metric space is continuous if and only if

$$\lim x_n = x \text{ implies } \lim g(x_n) = g(x) \tag{1}$$

and also if and only if

$$g^{-1}[G] \text{ is open whenever } G \text{ is an open interval.} \tag{2}$$

To check (1) for f one needs the triangular inequality; to check (2) one does not, namely, $f^{-1}[(a, b)] = \{x \mid a < d(x, z) < b\} = N_b(z) \sim D_a(z)$ is open. (Here, $N_r(z), D_r(z)$ are the open and closed spheres of radius r; center z.)

This inspires the following joke. Let X be a set and d a real function of two variables in X satisfying $d(x, x) = 0$ and $d(x, y) = d(y, x) > 0$ if $x \neq y$. Give X the topology which has as subbase the set of all $N_r(x)$ for all x and all $r > 0$. Fix z and define $f(x) = d(x, z)$.

THEOREM. (i) *f is continuous*, (ii) *f is not necessarily continuous.*

Proof. Using (2) as the criterion for continuity, the above proof yields continuity of f, proving (i). Now it is well known that for arbitrary topological spaces (2) implies (1) (although the converse is false). But a counterexample to (1) is given by taking X to be the positive reals and $d(x, y) = xy$ for $x \neq y$.

HOW ASYMMETRIC IS A PARALLELOGRAM?

FRED KRAKOWSKI, University of California, Davis

The asymmetry of a parallelogram π can be measured by the quotient $k(\pi) = A(\sigma)/A(\pi)$, where $A(\pi)$ is the area of π and $A(\sigma)$ denotes the area of the largest mirror-symmetric region σ contained in π. Clearly $k(\pi) \leqq 1$ and $k(\pi) = 1$ only if π is mirror-symmetric, *i.e.*, a rectangle or a rhombus. In this note we show that for all parallelograms $k(\pi) \geqq 2\sqrt{2} - 2$ or, in other words, every parallelogram contains a mirror-symmetric region which covers more than 82% of its area. The following elementary argument yields this result.

First observe that there is always a pair of straight lines x and y passing through the center of π which meet at right angles and divide π into four parts of equal area [1]. Assume that π is not a rhombus and let A, B, C, D be the vertices of π and X, Y, X_1, Y_1 the intersections of x and y with the sides AB, BC, CD, DA respectively (Fig. 1). As the four diagonal triangles of π are all equal in area, each side of π contains exactly one of the points X, Y, X_1, Y_1 in its interior. An affine transformation t with fixed line y such that it moves all points in the direction perpendicular to y also leaves x invariant, does not destroy symmetry about x and leaves ratios of area unchanged. If the ratio of t is chosen to be equal to $(YY_1)/(XX_1)$ and if by A', B', π', $\cdots$ we denote the

images of A, B, π, $\cdots$ under t, we then have $Y=Y'$, $Y_1=Y_1'$, $X'X_1'=YY_1$, $X'Y_1=X'Y$, and $\sphericalangle YX'Y_1=90°$. Hence, denoting by α, β, γ and α_1, β_1, γ_1 the interior angles of the triangles $\Delta X'A'Y_1$ and $\Delta X'B'Y$ respectively, we get the relations: $\alpha_1 = 90° - \alpha$, $\beta_1 = 180° - \beta$, $\gamma_1 = 90° - \gamma$.

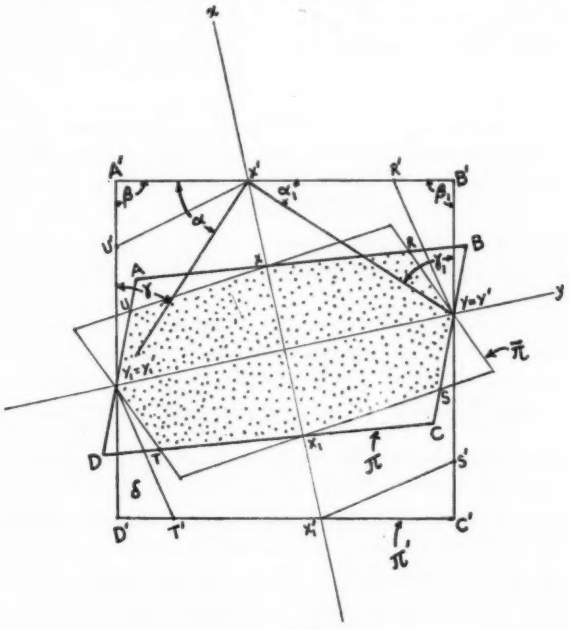

FIG. 1

Furthermore, the triangles $\Delta X'A'Y_1$ and $\Delta X'B'Y$ have the same area. Thus:

$$A'X' \sin\alpha = B'X' \sin\alpha_1 = B'X' \cos\alpha,$$

$$A'Y_1 \sin\gamma = B'Y \sin\gamma_1 = B'Y \cos\gamma,$$

$$A'X' \cdot A'Y_1 \sin\beta = B'Y \cdot B'X' \sin\beta_1 = B'Y \cdot B'X' \sin\beta.$$

From this we find $\tan\alpha \tan\gamma = (B'X' \cdot B'Y)/(A'X' \cdot A'Y_1) = 1$ and hence $\alpha+\gamma = 90°$, $\beta=\beta_1=90°$. We further observe that $\Delta X'A'Y_1=\Delta X'B'Y$ and so $A'B' = A'X'+X'B' = B'Y+A'Y_1 = B'Y+YC' = B'C'$. Thus it follows that the image π' of the parallelogram π is a square.

It might be interesting to note at this point that the lines x and y turn out to be the principal axes of the ellipse which has the diagonals of the given parallelogram as conjugate diameters, because t transforms this ellipse into a circle, the diagonals into a pair of orthogonal diameters and leaves x and y invariant.

Let ω now be the symmetric octagon $XRYSX_1TY_1U$ which encloses the area common to π and the mirror-image $\bar{\pi}$ of π with respect to x. Clearly $A(\omega)/A(\pi) = A(\omega')/A(\pi')$ and $A(\omega') = A(\pi') - 4A(\delta)$, where δ is one of the four congruent right triangles lying inside π' and outside ω'. Let a be the side of the square π' and u, v the legs of δ. The perimeter of δ is equal to $u+v+\sqrt{(u^2+v^2)} = a$. Now among all right triangles with the given perimeter a the isosceles right triangle has the largest area, which is found to be $\frac{1}{4}a^2(3-2\sqrt{2})$. Hence $k(\pi) \geqq A(\omega)/A(\pi) = A(\omega')/A(\pi') = 1 - 4A(\delta)/A(\pi') \geqq 2\sqrt{(2)} - 2$.

Reference

1. R. Courant and H. Robbins, What is Mathematics?, New York, 1960, p. 318.

CORRECTION

A. A. Mullin, *An abstract formulation of a problem related to Goldbach's conjecture*, this MONTHLY, vol. 68, 1961, pp. 487–488. In line 3 of Definition 1.3, for "$\bigcap_{i \in I} M_i$" read "M_i^x".

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND
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OFFERINGS AND ENROLLMENTS IN MATHEMATICS

A Summary of an Office of Education Report*

EDITH S. TREUENFELS, Wisconsin State College, Stevens Point

A study of offerings and enrollments in science and mathematics in public high schools is of particular interest and importance at the present time. To which degree the nation can hope to meet the demand for scientists and mathematicians, essential for its security, can be partially gauged by the extent to which science and mathematics are studied in the high schools. The extent to which high school pupils study these subjects now indicates the degree of scientific literacy of tomorrow's citizenry. The effectiveness of the various programs and grants in support of and for the improvement of mathematics and science instruction should be reflected in larger enrollments and improved offerings.

A small decline reported in percent of total high school enrollment does not, however, necessarily indicate a decrease in the number of pupils taking a certain subject. Since the number of high school pupils is much larger now

* Kenneth E. Brown and Ellsworth S. Obourn. *Offerings and Enrollments in Science and Mathematics in Public High Schools 1958*. Superintendent of Documents, U. S. Government Printing Office, Washington, 1961.

than, say, in 1900, the number of pupils taking a certain subject now will be found much larger than 60 years ago.

The data of this study were received in 1958 in response to two* questionnaires sent by the Department of Health, Education, and Welfare to a random sample of about 20% of the public high schools which are listed in the U. S. Office of Education. About 92% of the schools questioned replied. Eighty-three percent (4228) of the science questionnaires sent out were usable upon return; so were 83.5% (4254) of the mathematics forms.

In addition to information about the total enrollment in each grade, 8 through 12, the forms sought information about offerings and enrollments in the various science and mathematics courses. Whether or not the curriculum is being revised was asked on both questionnaires. There is a place on both forms for a "less," "same," or "more" statement comparing the present emphasis on science and mathematics to that given by the school three years earlier. A similar question about the "trend in science enrollment" appears on the science questionnaire only.

In reply to the last question, 1.6% of the schools of all sizes† report a decrease, 27.4% an increase, which leaves about two-thirds of the schools which report little change in their enrollment trend. It is encouraging that 42.9% of the schools with a total enrollment of less than 100 pupils report an increasing trend.

To the question: "What is the present emphasis on science in your school as compared to 3 years ago?" 97% of the returns had an answer. Approximately two thirds indicated greater, one third unchanged, and only half a percent less emphasis.‡ The data allowed for the conclusion that the size of the school seemed to be an influencing factor: "From the smallest to the largest schools there was a progressively greater percent indicating more emphasis as compared to that of 3 years earlier" (p. 17).

The size of the school apparently influenced the offerings, too. Every science course was offered by higher percentages of schools with enrollments of more than 200 pupils. The percentage of increase in the schools which offer science has been greater for chemistry and physics than for general science and biology.

It is important to use utmost care and caution in making generalizations from this study about conditions in all schools in the country. The questions about enrollment and offerings asked for information on the first term 1958–59, only. Some courses, particularly in mathematics, may have been offered the second semester and therefore are not recorded. Besides, the study makes use of enrollments in the grades where a certain course is most commonly offered; *i.e.*, general science in the 9th grade, biology in the 10th. The account may also

* On page 77, three questionnaires are mentioned as reproduced in the appendix. Only two are found there.

† These categories of school sizes were used: 1–99, 100–199, 200–499, 500 or more pupils.

‡ How "emphasis" is measured or how it manifests itself is not revealed.

conceal the true picture in the categories of smaller schools. There general science and biology, or chemistry and physics may be offered in alternate years, and consequently are not necessarily taken in the grade in which the course is most commonly offered.

Forty percent of the returns indicated that the mathematics curriculum was being revised. In 97.4% of the returned mathematics questionnaires the principals had answered the question on emphasis. Less than 1% reported decrease. Approximately 33.5% found that the emphasis remained the same, while more than 63% indicated an increase.

The enrollment in the grade where a certain mathematics course was most commonly offered had been chosen as the base to which to compare the enrollment in that course. Schools in this study with enrollments of 500 or more offered their pupils a much better opportunity to study mathematics than smaller schools. In some geographical regions, a smaller percent of the schools offered certain mathematics courses than others, so that "irrespective of cause, all pupils in this study did not have an equal opportunity to take mathematics" (p. 49).

Two facts are interesting to note, "The number of boys exceeded the number of girls in all the mathematics courses of this study" (p. 64); and, considering average class size, "At the very time when the pupils needed individual classroom assistance most to understand basic mathematical principles, they unfortunately found themselves in large classes" (p. 71).

General mathematics is considered to be a 9th-grade course. About 60% of the schools with 9th-grade enrollment offered general mathematics. Thirty-four and four tenths percent of the 9th-grade pupils in the study were enrolled in general mathematics. Twenty-nine and five tenths percent of all the pupils in this study were enrolled in schools not offering it. Fifty-three and five tenths percent of the enrollment in general mathematics were boys. The average class size was 26.5.

Elementary algebra was taken by 71.6% of the 9th-grade pupils. In general, only schools with very small enrollment did not offer elementary algebra. Only 1.5% of all the 9th-grade pupils in the study were in schools which did not offer it. In the 9th-grade algebra courses 53.2% were boys. The average class size was 27.4 pupils.

Plane geometry may be given in alternate years and is probably offered by a larger percentage of schools with 10th grade enrollment than the 84.3% which the study indicates. Forty-six and seven tenths percent of the 10th-grade pupils in the study were taking plane geometry, while 5.6% of them were in schools not offering this subject. Fifty-eight and three tenths percent of the plane geometry enrollment were boys. The average class size was 24.6 pupils.

Intermediate algebra or advanced algebra (presumably courses of the same content) is offered in the 11th grade by 73% of the schools. Of the 11th-grade pupils, 37.0% took intermediate or advanced algebra. Nine and seven tenths percent of all 11th-grade pupils in the study were not offered this opportunity

in their schools. Of those pupils taking it, 61.8% were boys. The average class size was 23.5 pupils.

The practice of offering trigonometry and solid geometry in alternate semesters seems to lose ground. A trigonometry-advanced algebra combination seems to gain favor. Solid geometry, when not offered as a separate course and if taught at all, is either integrated in plane geometry or treated for some weeks during the plane geometry course. Thus only 13.7% of the schools report offering a course in solid geometry. These had enrolled 3.9% of all the 12th-grade pupils in this study. Seventy-two and four tenths percent of the 12th-graders are in schools not offering it in the fall of 1958. Of those pupils taking solid geometry, 78.4% were boys. The average class has 17.5 pupils.

Forty-one and eight tenths percent of those schools which had 12th-grade enrollment offered trigonometry. Eleven and five tenths percent of the 12th-grade pupils took advantage of the offerings, while 33.2% were in schools not offering trigonometry. Seventy-six and four tenths percent of the trigonometry pupils were boys. The average class size was 17.

Three and eight tenths percent of the schools with a 12th grade offered a college mathematics course for advanced standing. Seventy-one and seven tenth percent of the pupils enrolled were boys.

An advanced general mathematics course for 11th- and 12th-grade non-college-bound pupils was reported by 16.2% of the schools; 55.8% of the enrollment were boys.

The study assumes with most of us that a larger enrollment, more offerings, and greater emphasis on science and mathematics courses are healthy, encouraging signs of a trend to the better in public education. The following uneasiness about such an unqualified assumption is of course not found in the study. In deed, an expression of such misgivings would not have a place in a statistical report. Nevertheless, a question arises which the study does not and should not attempt to answer: What areas of study, if any, have a smaller enrollment as a consequence of the increase in the science-mathematics enrollment? Has emphasis on literature, history, and the fine arts been sacrificed, thus over-emphasizing mathematics and science? Or have the schools managed a sound balance? How do the so-called nonacademic courses like typing and driver education fare? Are summer-school sessions utilized to teach these and other skills or is the student with a four-year science-and-mathematics program being deprived of training in these skills he needs now more urgently than ever? In other words: Will tomorrow's college freshman be well educated? Or will he be a specialist, literate in his subject only?

A GENERATION OF HIGH SCHOOL CALCULUS*

J. H. NEELLEY, Carnegie Institute of Technology

For the past four years we have had freshmen come to Carnegie Institute of Technology with some high school calculus. We have tried to meet the desires of these freshmen in several ways. The first group was integrated with second-semester students. This was very unsatisfactory. The following three groups have been kept apart and have been given a course in calculus and analytics to cover four semesters in two. This course has varied some in the amount of analytics since a full course proved to be too much.

I have watched carefully these accelerated students and am now able to report on our first college generation of such!

In 1956, the press, politicians and then the public began to demand that the high schools do the work equivalent to that done in Europe under a very different situation. This did create the high school accelerated programs. So, in 1957–58, we had 13 freshmen register as majors in mathematics, whereas before then we had had less than 5 each fall. This pressure has gradually increased our mathematics majors to 32 in 1958–59, 37 in 1959–60 and this year to 46 for 1960–61.

In these four years the freshman classes have had an increased number of accelerated high school students. Only 6 in 1957–58, 48 in 1958–59, 33 in 1959–60 and 54 in 1960–61.

These are our brightest students and so it is interesting to see how they have registered in college. The first year, 2 of the 6 registered for mathematics as major. The second year 5, then the third year 8 and this year 5.

These figures are surprising. When only 5 out of 54 register for mathematics as major when at the same time we have 46 so register from the freshman class, we are shocked. This carries on the whole four years, 2 of 13, 5 of 32, 8 of 37 and 5 of 46. So our mathematics majors are not coming in large numbers from the high school accelerated groups.

Also it is interesting to see how the accelerated group holds its own in college. The first group of 2 dropped to 1 in short order. This is the year we had 13 mathematics majors. Today this class is the senior and the pressure has made it grow from 13 to 20 with only 1 accelerated student.

The 1958–59 group started at 48 and only 11 completed the first year of college work. There were 5 who originally registered as mathematics majors and today, as juniors, only 3 of them are still such. This is 3 out of 25 juniors today.

The 1959–60 group started at 33 and 18 finished the first year of college mathematics. This course was cut some as compared to the 1958–59 one. Today we have 31 sophomores who are mathematics majors and only 4 of that original group are still such.

* Presented to the Mathematics Council of Western Pennsylvania at Annual Conference, March, 1961, and to the Mathematical Association of America, May, 1961.

Our fourth year, 1960–61 saw 54 freshmen come with high school calculus and today we have 46 freshmen mathematics majors with only 5 of them from the accelerated high school group. The 54 is now down to 17 and the first year is not completed.

To summarize the generation, out of 1700 engineering and science freshmen, 141 came to us with high school calculus. As of now, we have 14 or only 10 per cent of that group as mathematics majors. This is even more striking in that we now have 122 mathematics majors in Tech.

These figures seem surely to point to two things. First, high school accleration is not the way to increase our number of college majors in mathematics. Second, the high casualty in the accelerated college courses make it seem that "high school calculus is largely a waste of time."

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1491. *Proposed by J. W. Andrushkiw, Seton Hall University*

Show that there exists at least one and at most three equilateral triangles inscribed in a parabola and having a given point P on the parabola for a vertex.

E 1492. *Proposed by J. L. Brown, Jr., Ordnance Research Laboratory, Pennsylvania State University*

Considering unity as a prime, show that every positive integer can be written as a sum of distinct primes.

E 1493. *Proposed by L. Flatto, N. C. Hsu, and A. G. Konheim, IBM Research Center, Yorktown Heights, New York*

Prove that a nonzero integral polynomial $h(x)$ (*i.e.*, one with integer coefficients) which vanishes at $x=1$ and $x=2$ must have a coefficient $\leqq -2$.

E 1494. *Proposed by George Sadowsky, Combustion Engineering, Inc., Windsor, Connecticut*

Invert the matrix $[a_{ij}]=[x^{|i-j|}]$, $x\neq 0, \pm 1$.

E 1495. *Proposed by Dunstan Hayden, The Priory School, Washington, D. C.*

Let $f(z)=z^2$, and let the domain of this function be any straight line in the

complex plane. The graph of the range of f is then a parabola in the complex plane. What are the coordinates of its vertex?

SOLUTIONS

A Doubly True Addition

E 1461 [1961, 378]. *Proposed by Underwood Dudley, University of Michigan*

Solve the cryptic addition

$$\begin{array}{r} \text{F I F T Y} \\ \text{F O U R} \\ \text{F O U R} \\ \text{T W O} \\ \hline \text{S I X T Y} \end{array}$$

remembering that FOUR+12 is a perfect square.

I. *Solution by C. W. Trigg, Los Angeles City College.* Replace the letter O with θ. S>F≠9, so F is 4 or 5, whereupon S=F+1. Since $2R+\theta=10$ or 20, the only possible value of $F\theta UR(=n^2-12)$ is 4612. Then S=5 and W=10−2U−1=7. Now $F+2\theta+T+1=X+20$, T=X+3, and X=0, T=3. Finally, (I, Y)=(8, 9) or (9, 8). Thus there are two of these doubly true additions:

$$48439 + 4612 + 4612 + 376 = 58039, \quad 49438 + 4612 + 4612 + 376 = 59038.$$

II. *Solution by J. A. Lambert, Newcastle University College, N.S.W., Australia*

We have the congruences (modulo 10)

(1) $\qquad R + R + O \equiv \text{zero},$

(2) $\qquad U + U + W + (1, 2, \text{or } 3) \equiv \text{zero},$

(3) $\qquad F + O + O + T + (1, 2, \text{or } 3) \equiv X,$

(4) $\qquad F + F + (1, 2, \text{or } 3) \equiv \text{zero},$

the equation

(5) $\qquad F + (1 \text{ or } 2) = S,$

and

(6) $\qquad$ FOUR + 12 is a perfect square.

Now (4) shows that F=4 or 9; (5) that F=4, S=5; (6) gives (by examination of perfect squares) 4612 for FOUR; (2) now gives W=7; (3) leads to T=3, X=zero. I and Y are not uniquely determined; each might be either 8 or 9.

Also solved by A. N. Aheart, Ronald Alter and Herbert Gintis (jointly), Bonnie Amner, M. H. Auerbach, J. W. Baldwin, Merrill Barnebey, Robert Bart, Charles Bartha, H. F. Bennett, C. R. Berndtson and E. R. Williams (jointly), Jeanette Bickley, T. P. Bleakney, Walter Bluger, A. Bollobás, Christopher Boorse, Robert Bowen, K. C. Bower, Brother Patrick Ronald, Brother Louis Zirkel, Robert Budny, W. E. Buker, Bradford Burdick and Robert Spira (jointly), F. P.

Callahan, Jr., Robert Carlos, Charles Carniglia, F. H. Cleveland, D. I. A. Cohen, E. L. Cohen and G. M. Leibowitz (jointly), R. J. Cormier, J. B. Deeds, Monte Dernham, D. Drasin, J. R. Durbin, E. S. Eby, A. D. Egendorf, Aaron Eidelman, Jane Evans, J. A. Fauches, F. E. Fischer, David Forslund, C. S. Patlak and Seymour Geisser (jointly), Howard Givner, Anton Glaser, Larry Glickfeld, Robert Goldberg, Michael Goldberg, L. D. Goldstone, Jay Gottesfeld, R. G. Green, Corinne Hattan, F. W. Herlihy, Margaret Herzog, J. E. Homer, Jr., Alice Hunt, J. A. H. Hunter, C. R. Hutchinson, A. R. Hyde, Daniel Isaacson, Diane M. Johnson, William Kantrowitz, Harry Kapper, Jr., Leonard Klosinski, Kenneth Kloss, Donald Knuth, R. R. Korfhage, Sidney Kravitz, Harry Langman, Dean Lawrence, H. L. Lawton and Anna H. Watson (jointly), H. R. Leifer, R. J. Lewyckyj, William Lopez, James Lucke, David McCarroll, W. M. McKeeman, D. C. B. Marsh, T. H. Means, G. J. Michaelides, J. W. Milsom, Otto Mond, D. A. Moran, Mr. and Mrs. D. L. Muench (jointly), J. B. Muskat, Herbert Nadler, H. L. Nelson, H. J. Noble, C. S. Ogilvy, E. A. Passow, Sidney Penner, Walter Penney, L. B. Perry, D. J. Persico, J. P. Phillips, J. L. Pietenpol, C. F. Pinzka, Harsh Pittie, M. T. Rincon, L. A. Ringenberg, O. J. Roman, David Sachs, H. Schloss, L. J. Schneider, Mitchell Secondo, S. J. Sidney, D. R. Simpson, Carl Spitznagel, E. L. Spitznagel, Jr., W. B. Stovall, Jr., Eric Sturley, Paul Stygar, G. C. Thompson, Guy Torchinelli, Donald Vogel, W. C. Waterhouse, Alan Weinstein, Charles Wexler, Sam Wheatman, D. R. Wilder, R. H. Wilson, Jr., R. L. Yates, Walter Zayachowski, and the proposer. Late solutions by Rafael Gallego-Diaz, Roger Hunt, James Johnson, L. J. Katz, Jr., Adolph Lu, Wayne McPherson, B. B. Mapinin, Soran Stojakovič, and Rolland Sturtevant.

Among the suggestions to render the solution unique were: (1) add the condition "YOU is a square" or "I is a square" (Pinzka), (2) add the condition "FIFTY and SIXTY are even" (Marsh), (3) add the condition "SIX+7 is prime" (Bleakney and Pietenpol), (4) replace "FOUR+12 is square" by "FIFTY+39 is prime" (Knuth).

Hunter, who has originated the name "alphametic," suggested the following examples

$$\begin{array}{r} \text{SIX} \\ \text{SEVEN} \\ \hline \text{SEVEN} \\ \hline \text{TWENTY} \end{array} \qquad \begin{array}{r} \text{SIX} \\ \text{SIX} \\ \hline xxxx \\ xxxx \\ xxAS \\ \hline \text{ADOZEN} \end{array}$$

where in the second one the x's indicate digits. The proposer offered the following additional alphametics: TWENTY+FIVE+FIVE=THIRTY and WHEN is a square, SEVEN+THREE+FIVE+FIVE+TEN=THIRTY.

Nelson pointed out that TEN+TEN+FORTY=SIXTY has a unique solution if the base of the arithmetic is specified to be 10; if the base is not specified, there are infinitely many solutions. He then proposed showing that each of the two cryptic divisions

$$\begin{array}{r} \text{IA} \\ \text{EEL})\text{MAIL} \\ \text{EEL} \\ \hline \text{IMUL} \\ \text{ISMS} \\ \hline \text{EU} \end{array} \qquad \begin{array}{r} \text{WE} \\ \text{SEE})\text{SOOT} \\ \text{SAWN} \\ \hline \text{NEAR} \\ \text{NNNW} \\ \hline \text{NYC} \end{array}$$

has a solution for only one radix, and then the solution is unique.

For other cryptarithms and arithmetical restorations the reader might consult problems 3212, E1, E7, E10, E20, E22, E23, E30, E37, E43, E58, E71, E78, E99, E105, E130, E140, E160, E164, E173, E184, E192, E198, E217, E251, E258, E265, E297, E317, E333, E751, E891, E971, E1110, E1111, E1241.

Alternating Colors

E 1462 [1961, 378]. *Proposed by Michael Skalsky, Southern Illinois University*

In how many ways can 4 white, 3 black, and 3 red balls be arranged in a row so that no two adjacent balls are of the same color?

Solution by C. F. Pinzka, University of Cincinnati. Consider the $6!/3!^2=20$ arrangements of the black and red balls only. Suppose that one of these arrangements has r balls following balls of the same color. The white balls may be inserted in $\binom{7-r}{4-r}$ ways to give arrangements in which no two adjacent balls are of the same color. The total number of arrangements of this kind is then $\sum_{r=0}^{4} a_r\binom{7-r}{4-r}$, where a_r is the number of arrangements of the black and red balls in which r balls follow balls of the same color. By direct enumeration it is found that $a_r=2, 4, 8, 4, 2$ for $r=0, 1, 2, 3, 4$ and the desired number of arrangements is 248.

Also solved by A. N. Aheart, J. W. Baldwin, H. F. Bennett, Walter Bluger, Robert Bowen, Brother Alfred, Brother Louis Zirkel, W. E. Buker, Charles Carniglia and Alan Felzer (jointly), V. D. D'Antonio and R. C. Potter (jointly), Monte Dernham, Underwood Dudley, Jane Evans, Seymour Geisser and C. S. Patlak (jointly), Howard Givner, Michael Goldberg, L. D. Goldstone, Jay Gottesfeld, A. S. Gregory, F. W. Herlihy, A. R. Hyde, D. M. Johnson, Erwin Just, Francis Katcher and R. T. Shannon (jointly), Donald Knuth, Jiang Luh, David McCarroll, Daniel Maki, D. C. B. Marsh, Walter Penney, Mary Agnes Racki, L. A. Ringenberg, David Sachs, L. J. Schneider, S. J. Sidney, W. B. Stovall, Jr., Guy Torchinelli, and Walter Zayachkowski. Late solutions by Daniel Isaacson, J. B. Muskat, and Mirko Stojaković.

Fourteen of these solutions disagreed with 248 as the final answer.

Geisser and Patlak arrived at the solution by inserting the appropriate values into equation 4.2 (p. 375) of A. M. Mood, "The Distribution Theory of Runs," *Ann. Math. Stat.*, 11, 1940, pp. 367–392. Goldstone called attention to Probs. 88, 89, 97, 98, 100, 170, 724, 725 in Whitworth's *Choice and Chance.*

A Locus in Three-Space

E 1463 [1961, 378]. *Proposed by V. F. Ivanoff, San Carlos, California*

Given an imaginary point P: $(a+pi, b+qi, c+ri)$ in 3-space. Find the locus of real points (x, y, z) whose distance from P is real.

Solution by Underwood Dudley, University of Michigan. We must have

$$(a-x+pi)^2+(b-y+qi)^2+(c-z+ri)^2$$

real and nonnegative, or $p(a-x)+q(b-y)+r(c-z)=0$ and

$$(a-x)^2+(b-y)^2+(c-z)^2 \geqq p^2+q^2+r^2.$$

Thus the locus consists of the points on the plane through (a, b, c) perpendicular to the direction (p, q, r) and not interior to the circle on the plane with center (a, b, c) and radius $(p^2+q^2+r^2)^{1/2}$.

Also solved by C. B. Barfoot, Merrill Barnebey, Robert Bart, H. F. Bennett, A. Bollobás, Brother Louis Zirkel, E. L. Cohen, J. R. Durbin, Jane Evans, David Friedman, Seymour Geisser

and C. S. Patlak (jointly), Michael Goldberg, Robert Goldberg, L. D. Goldstone, Carl Harris, A. R. Hyde, Erwin Just, J. J. Kim, Harry Langman, Jiang Luh, W. M. McKeeman, D. C. B. Marsh, D. A. Moran, D. L. Muench, Amos Nannini, Walter Penney, D. J. Peterson, J. L. Pietenpol, C. F. Pinzka, David Sachs, S. J. Sidney, Barry Simon, Paul Stygar, Kay Leon Tjio, Guy Torchinelli, W. C. Waterhouse, Walter Zayachkowski, and the proposer.

A number of these solutions were not complete.

Both McKeeman and the proposer pointed out that the problem is easily extended to n-dimensional space.

An Application of the Arithmetic-Geometric Inequality

E 1464 [1961, 378]. *Proposed by Freddy Storey, Princeton University*

Show that for $n \geq 2$, $\prod_{i=0}^{n} \binom{n}{i} \leq \{(2^n-2)/(n-1)\}^{n-1}$.

Solution by David Zeitlin, Remington Rand Univac, St. Paul, Minn. Since

$$\prod_{i=0}^{n}\binom{n}{i} = \prod_{i=1}^{n-1}\binom{n}{i}, \qquad 2^n - 2 = \sum_{i=1}^{n-1}\binom{n}{i},$$

the desired inequality is merely the arithmetic-geometric inequality for the $n-1$ numbers $\binom{n}{i}$, $i=1, \cdots, n-1$.

Also solved by A. N. Aheart, D. W. Bailey, A. Bollobás, D. A. Breault, J. L. Brown, Jr., Leonard Carlitz, David Chale, A. J. Chandy, J. L. Cline, D. I. A. Cohen, J. B. Deeds, G. C. Dodds, Underwood Dudley, Sarah Evangelista and Peter Hagis, Jr., (jointly), H. M. Feldman, J. H. Folkman, David Forslund, David Friedman, Michael Goldberg, A. S. Gregory, J. C. Hickman, A. R. Hyde, Erwin Just, Leonard Klosinski, Kenneth Kloss, J. J. Kim, Hsü-Tung Ku, Viktors Linis, Jiang Luh, W. M. McKeeman, P. B. Manchester, D. C. B. Marsh, J. B. Muskat, D. J. Persico, C. F. Pinzka, Harsh Pittie, David Sachs, Norman Schaumberger, S. J. Sidney, T. H. Slook, Evelyn Strawbridge, Guy Torchinelli, W. C. Waterhouse, and the proposer. Late solution by B. B. Mapinin.

Carlitz showed, in addition, that $\ln \prod_{i=0}^{n}\binom{n}{i} = n^2/2 + O(n \ln n)$.

A Non-Biplanar Graph

E 1465 [1961, 379]. *Proposed by U. R. Kodres, IBM Corp., Poughkeepsie, New York*

It is well known that the graph which represents the classical problem of connecting three houses to three utilities is not planar. Prove that a generalization of this graph, namely the graph which represents connecting seven houses to seven utilities is not biplanar, *i.e.*, the graph cannot be factored into two planar factors.

Solution by D. C. B. Marsh, Colorado School of Mines. If the seven-to-seven graph were biplanar, then one of its factors would have to contain at least 25 arcs. However, an n-to-n set may be (partially) connected by at most $4(n-1)$ coplanar arcs which are nonintersecting. When $n=7$, this yields a maximum of 24, showing the nonexistence of a biplanar graph in this case.

Also solved by David Sachs and the proposer.

Editorial Note. A similar proof shows that the problem of connecting $4m-1$ houses to $4m-1$ utilities is not m-planar.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Bloomfield College

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Bloomfield College, Bloomfield, N. J. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4995. *Proposed by Oystein Ore, Yale University*

When $\phi(x)$ denotes Euler's function it is readily verified that $n=14$ is the smallest even number such that the equation $\phi(x)=n$ has no solution. Prove that for each exponent α there is a smallest odd integer k_α such that the equation $\phi(x)=2^\alpha k_\alpha$ has no solution. Determine k_2, k_3, k_4. Try to find bounds for k_α.

4996. *Proposed by Richard Bellman, RAND Corporation, Santa Monica, Calif.*

Let $\{x_n\}$ be a sequence of nonnegative N-dimensional vectors and let A be an N by N nonnegative matrix. If $x_{n+1} \leqq Ax_n - y_n$ (in the sense of inequality component by component), and if y_n is nonnegative with $\sum_{n=1}^{\infty} y_n < \infty$, under what conditions on A does the sequence $\{x_n\}$ converge?

4997. *Proposed by Donald Newman, Yeshiva University*

Show that the power series of $e^{(z+1)/(z-1)}$ has coefficients $O(n^{-3/4})$, but not $o(n^{-3/4})$.

4998. *Proposed by Meyer Wolf, University of Minnesota*

S. H. Unger has presented a proof concerning the self-overlapping of cyclically equivalent sequences (This MONTHLY, Feb. 1960, pp. 139–143), that depends on the property of being made up of a chain of identical subsequences. Prove that the composition of the sequence formed from the concatenation of two sequences X and Y is independent of the order of concatenation if and only if X and Y are chains of a particular subsequence Z.

4999. *Proposed by R. B. Deal, Oklahoma State University*

Is there a finitely additive probability on the set of all subsets of the positive integers which assigns probability zero to all finite sets?

5000. *Proposed by R. C. Lyness, Singleton Lodge, near Blackpool, England*

$x^3-2x^2+2x-1=0$ has roots which are all roots of unity and so $w_{n+3}=2w_{n+2}-2w_{n+1}+w_n$ with $w_0=$ is a cubic recurrence with an infinite number of zeros, w_{6n}.

$2x^3-2x^2+2x-1=0$ gives the recurrence $w_{n+3}=w_{n+2}-w_{n+1}+\frac{1}{2}w_n$ and with

$w_0 = w_1 = 0$ we have $w_m = 0$ for $m = 0, 1, 4, 6, 13, 52$.

Can a cubic recurrence with only a finite number of zeros have more than six?

SOLUTIONS

Matrix with Eigenvalues 0 and 1

4872 [1959, 817]. *Proposed by Ky Fan, Wayne State University*

Let A be a matrix (not necessarily square) of rank $r \geqq 1$ and with nonnegative elements. Let A^* denote the transpose of A. Prove that the square matrix AA^* has no eigenvalue different from 0, 1, if and only if, after deleting all identically vanishing rows and columns, the remaining submatrix of A can be brought by a permutation of the rows and a permutation of the columns to the form

$$B = B_1 + \cdots + B_r, \tag{1}$$

where, for each i, B_i is a rectangular matrix of rank 1, with all its elements positive and such that the sum of squares of all elements of B_i is 1. Equation (1) means that B is obtained by laying out successively the rectangular blocks $B_1, \cdots, B_r$ with the lower right corner of B_i attached to the upper left corner of B_{i+1}, and with zeros filling in the entire matrix outside these blocks.

Solution by Hans Schneider, the University of Wisconsin. If H is a nonnegative matrix whose maximal eigenvalue ρ is k-fold, then there exists a permutation matrix P, for which

$$PHP^* = H_1 \oplus \cdots \oplus H_k \oplus L,$$

where the H_i are irreducible square matrices having ρ as a simple eigenvalue, and the maximal eigenvalue of L is less than ρ. It follows that, if the eigenvalues of H are 1 or 0, then $k = r$, the rank of A. Thus

(a) If AA^* satisfies the given conditions then, for some permutation matrix P, $PAA^*P^* = H_1 \oplus \cdots \oplus H_r \oplus 0$, where each H_i is an irreducible matrix of rank 1 and trace 1; and (obviously) conversely.

If b_i and b_j are rows of a nonnegative rectangular matrix B, such that $b_i b_j^* = 0$, then no column of B has positive elements in both b_i and b_j. If $B_i B_i^*$ is irreducible then B_i has no zero rows. Hence

(b) If $A = H_1 \oplus \cdots \oplus H_k \oplus 0$, where the H_i are irreducible if and only if $B = PAQ = B_1 \oplus \cdots \oplus B_r \oplus 0$ for all permutation matrices Q, where the B_i are rectangular matrices (having the same number of rows as H_i), then $B_i B_i^* = H_i$, and B_i has no zero rows and, provided Q is properly chosen, no zero columns.

It is known that rank $B_i B_i^*$ = rank B_i, and the sum of squares of elements of B_i equals trace $B_i B_i^*$. Hence

(c) The matrix $H_i = B_i B_i^*$ is of rank 1 and trace 1, if and only if B_i has rank 1, and the sum of squares of its elements is 1.

(d) If a matrix of rank 1 has a zero element, then it has a zero row or col-

umn, and (again obviously) conversely.

Combining (a), (b), (c) and (d) we obtain the desired results.

Also solved by the proposer.

Distinct Sums in a Nonassociative Algebra

4931 [1960, 926]. *Proposed by Harry Goheen, Oregon State College*

In an algebra for which there is a commutative law but no associative law, how many formally distinct sums are there of n elements?

I. *Solution by J. R. Brown, University of Massachusetts.* Let the number of distinct sums of n elements be S_n. Such a sum must be composed of two subsums which are composed of k and $n-k$ elements respectively. By commutativity we have

$$S_n = \frac{1}{2}\sum_{k=1}^{n-1}\binom{n}{k}S_kS_{n-k}.$$

Hence, since $S_1=1$, and $\binom{n+1}{k}=\binom{n}{k}+\binom{n}{k-1}$,

$$\begin{aligned}S_{n+1} &= \frac{1}{2}\sum_{k=1}^{n}\binom{n+1}{k}S_kS_{n+1-k}\\ &= \frac{1}{2}S_n + \frac{1}{2}\sum_{k=1}^{n-1}\binom{n}{k}S_kS_{n+1-k} + \frac{1}{2}\sum_{k=2}^{n}\binom{n}{k-1}S_kS_{n+1-k} + \tfrac{1}{2}S_n\\ &= \frac{1}{2}\sum_{k=1}^{n-1}\binom{n}{k}[S_kS_{n+1-k} + S_{k+1}S_{n-k}] + S_n.\end{aligned}$$

Assume $S_{k+1}=(2k-1)S_k$ for $k<n$. Then

$$\begin{aligned}S_{n+1} &= \frac{1}{2}\sum_{k=1}^{n-1}\binom{n}{k}[(2n-2k-1)+(2k-1)]S_kS_{n-k} + S_n\\ &= (2n-2)S_n + S_n = (2n-1)S_n.\end{aligned}$$

Since $S_2=1=(2-1)S_1$, this last result is true for all n. Consequently, for $n\geqq 2$,

$$S_n = 1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3) = \frac{(2n-3)!}{2^{n-2}(n-2)!}.$$

R. W. Wagner has pointed out the close relation of this problem to 3954 [1941, 564–9].

II. *Solution by James Singer, Brooklyn College.* Since summation is a binary operation, $n-1$ pairs of parentheses must be used to render an indicated sum of n nonassociative elements meaningful (this includes the final pair of parentheses around the first and last elements). The $n-1$ pairs of parentheses can be placed in $\binom{2n-2}{n-1}/n$ ways. (See A. Cayley, *On the analytic forms called trees, Second Part*; Phil. Mag., vol. 18, 1959, pp. 374–378; also *Collected Works*, vol. 4,

pp. 112–115. The formula has been frequently rediscovered since then.) The n elements can be arranged in $n!$ ways to yield $(n-1)!\binom{2n-2}{n-1}$ sums. Since the two terms within any one pair of parentheses can be interchanged without changing the sum, the number of distinct formal sums is

$$\frac{(n-1)!}{2^{n-1}}\binom{2n-2}{n-1}.$$

Also solved by R. J. Cornier, S. H. Greene, Jiang Luh, B. J. M. Morselt, R. C. Read, and the proposer.

Solutions of a Fourth-Order Differential Equation

4932 [1960, 927]. *Proposed by D. S. Mitrinovitch, Belgrade University, Yugoslavia*

Consider the differential equation $z^2(d^4w/dz^4)=a(d^2w/dz^2)$, where a is a real parameter, $z=x+iy$. Determine the singularities of the complex functions $w=f(z, a)$ which satisfy it.

Solution by Emil Grosswald, University of Pennsylvania. The formal solution of the equation is straightforward. Denoting differentiation by accents and setting $w''=v$, the equation reads: $z^2v''=av$, of Euler-Cauchy type. According to the value of a, we distinguish the following cases:

(1) $a=0$, $v''=w''''=0$, whence $w=Az^3+Bz^2+Cz+D$;

(2) $a=-\frac{1}{4}$, $v=(C_1+C_2\log z)z^{1/2}$, whence $w=(A\log z+B)z^{5/2}+Cz+D$;

(3) $a=2$, $v=C_1z^2+C_2z^{-1}$, whence $w=Az\log z+Bz^4+Cz+D$;

(4) $a=6$, $v=C_1z^3+C_2z^{-2}$, whence $w=A\log z+Bz^5+Cz+D$.

If $a\neq 0$, $-\frac{1}{4}$, 2, 6, then $v=C_1z^s+C_2z^t$, where s and t are the distinct roots of $x^2-x-a=0$, and one obtains

$$(*) \qquad w = Az^{s+2} + Bz^{t+2} + Cz + D.$$

In particular, if a is of the form

(5) $a=r(r-1)>-\frac{1}{4}$, with rational $r=p/q$ (p, q coprime integers), then $s=r$, $t=1-r$ in (*); if

(6) $a>-\frac{1}{4}$ but $\neq r(r-1)$ with rational r, then s and t are distinct, real, irrational numbers; and finally, if

(7) $a<-\frac{1}{4}$, then s and t are complex conjugate numbers and (*) may be written as

$$w = z^{5/2}(A\sin(c\log z) + B\cos(c\log z)) + Cz + D,$$

with $c=\frac{1}{2}(4|a|-1)^{1/2}$.

Hence, excluding the trivial case $A=B=C=0$ when $w=D$ a constant, it follows that w has logarithmic branch-points at $z=0$ and $z=\infty$ in cases (2), (3), (4) if $A\neq 0$, and in cases (6), (7) if A, B are not both zero; these branchpoints become algebraic, of order one in case (2) if $A=0$, $B\neq 0$, and of order $q-1$ in case (5) with A and B not both zero. In the remaining cases, w is a polynomial

and its only singularity is the pole at $z=\infty$ of order equal to the highest non-vanishing power of z.

Also solved by G. DiAntonio.

Union of Commutative Fields

4933 [1960, 927; 1961, 299]. *Proposed by I. N. Herstein, Cornell University*

Suppose a ring R is the set-theoretic union of a finite number of commutative fields having the same unit element; prove that R must then be a commutative field.

I. *Solution by B. R. Toskey, Seattle University.* The common unit element must be the identity for R, and each element of R has an inverse, since it is in a field containing identity. Thus R is a division ring. Now, let n be the number of fields given, so that for any two elements a, x, in R, at least two of the elements a, ax, ax^2, $\cdots$, ax^n must lie in the same field and hence commute with each other. Thus $ax^kax^j=ax^jax^k$ for some k, j with $k>j\geqq 0$, and thus $x^{k-j}a=ax^{k-j}$. But, since $0<k-j\leqq n$, we have $x^{n!}a=ax^{n!}$. This shows that $x^{n!}$ is in the center of R for any x in R. By a theorem of Kaplansky [Canad. J. Math., vol. 3, 1951, pp. 290–292; also Jacobson, *Structure of Rings*, p. 185], R is therefore a commutative field.

II. *Solution by Robert Spira, Berkeley, California.* Let $R=\bigcup_{i=1}^{n} F_i$ where the F_i are commutative fields. Let j be such that $F_k\subset\bigcup_{i=k+1}^{n} F_i$ for $k<j$, but F_j is not contained in $\bigcup_{i=j+1}^{n} F_i$. As this is a finite union, such a j must exist. Set $B=\bigcup_{i=j+1}^{n} F_i$. Clearly $R=\bigcup_{i=j}^{n} F_i$. It will be shown that $R=F_i$ for one of the F_i's in this last union.

For if not, we can take $b\in B$, $b\notin F_j$ (as $F_j\neq R$); and also we can take $a\in F_j$ and $a\notin B$, by the defining property of F_j. The product ab lies either in F_j or in B. If $ab\in F_j$, multiply by a^{-1}, obtaining $a^{-1}ab=b\in F_j$, a contradiction (using here the fact that the units of the fields coincide). If $ab\in B$, multiply by b^{-1}, obtaining $abb^{-1}=a\in B$, a contradiction. Thus the product ab does not lie in R, contradicting assumption that R is a ring. Thus R must be one of the given F_i's, and is hence trivially a commutative field.

Also solved by R. M. Cohn and E. R. Gentile, Carl Faith, Melvin Henriksen, R. B. Kirchner, H. B. Mann, Barbara L. Osofsky, W. Peremans, E. C. Posner and Neal Zierler, W. R. Scott, R. T. Shannon, Jack Silver, K. R. Unni, William Veeck and Joan Richardson, Seth Warner, W. C. Waterhouse, and the proposer. Late solutions by Gerald Janusz and W. S. Martindale.

Editorial Note. The proposition as originally stated (omitting reference to a common unit) is not correct. The direct sum of two or more copies of the prime field Z_2 of characteristic 2 is not a field but it is a union of fields. Cohn and Gentile prove: *A ring which is a union of a finite number of fields is either a field or a direct sum of (a finite number of) copies of* Z_2.

Henriksen notes that Biatynicki-Birula, Browkin, and Schinzel have shown that a field cannot be the union of finitely many proper subfields (Colloq. Math., vol. 7, 1959, pp. 31–32).

Group Algebra

4934 [1960, 927]. *Proposed by I. N. Herstein, Cornell University*

Let F be a field of characteristic 0, let G be a group and $\Gamma(G, F)$ the group algebra of G over F. Prove that $\Gamma(G, F)$ is an algebraic algebra over F if and only if G is locally finite.

Solution by E. C. Posner, Harvey Mudd College, and Neal Zierler, Massachusetts Institute of Technology. Suppose G is locally finite and let $x = \sum_{i=1}^{n} a_i g_i$ be a member of Γ where $a_i \in F$ and $g_i \in G$. The subgroup H of G generated by $g_1, \cdots, g_n$ is finite by hypothesis, so its group algebra over F, which contains x, is a finite dimensional vector space over F. Hence there exists $m > 0$ such that x^{m+1} is linearly dependent on $x, \cdots, x^m$, *i.e.*, Γ is algebraic.

Conversely, suppose Γ is algebraic, let $g_1, \cdots, g_n$ be elements of G and let $x = g_1 + \cdots + g_n$. It follows from the hypothesis that for some $m > 0$, x^{m+1}, $x^{m+2}, \cdots$ are all linearly dependent on $x, \cdots, x^m$. But every product of i factors from among $g_1, \cdots, g_n$ appears with nonzero coefficient in x^i, $i = 1, 2, \cdots$, since the characteristic of F is 0, and hence, by the preceding statement, every product of more than m factors is equal to a product of m or fewer.

Also solved by D. B. Coleman, Barbara Osofsky, and the proposer.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. A. Rosenbaum, Department of Mathematics, Wesleyan University, Middletown, Connecticut, and not to any of the other editors or officers of the Association.

Introductory Analysis. By V. O. McBrien. Appleton-Century, Crofts, New York, 1961. 188 pp. $4.50.

The author sets himself the modest task of informing the reader about some fundamental concepts and ideas of mathematics and illustrating their utility in various sciences. The text deals with such concepts as sets, sequences, functions and their derivatives and integrals. The author gives no proofs, but he tries to state precise definitions—there is a three-page index of them. Unfortunately some of them are incomplete or incorrect or superfluous. This is particularly true of the chapters on limits and on the Riemann integral. These defects do not seriously detract from the challenging character of the text. There is a good list of references at the end of each chapter that will surely intrigue the good student; how well the book will serve the average student is problematical.

M. S. KNEBELMAN
Washington State University

Analytic Geometry with Calculus. By R. C. Yates. Prentice-Hall, Englewood Cliffs, N. J., 1961. xi+247 pp. $5.95.

This book is really a textbook in analytic geometry intended to prepare students for calculus. The basic ideas of analysis (real numbers, variables, functions, limits, continuity, derivative, and antiderivative) are concisely covered in the first 65 pages and then are used to support the study of analytic geometry of both two and three dimensions.

Direction cosines and direction numbers are introduced early. Curve sketching and polar coordinates are emphasized. The inverse problem of finding the equation of a curve from known properties leads to the area measure function, or antiderivative. The symbol "$\int$" does not appear anywhere in the text.

The conics are defined in terms of eccentricity, and then the distance properties are derived for each case. The crossed parallelogram linkage is shown as a generator of both the ellipse and the hyperbola. Two topics of special interest are LORAN, and the creation of the conics by paper folding. The chapter, "Some Mechanical Motions and Loci," is excellent. Clear sketches of the linkages accompany the analyses for ellipses, conchoids, cycloids, limaçons, lemniscates and cissoids. Elliptical gears and hyperbolic gears are diagrammed.

There is a brief but clear presentation of affine linear transformations. Matrices are introduced to characterize rotations, similitude, and stretch. The Peaucellier linkage illustrates inversion.

LAUREN G. WOODBY
Central Michigan University

Introduction to Geometry. By H. S. M. Coxeter. Wiley, New York, 1961. xiv+443 pp. $9.95.

The announced purpose of this book is an attempt to revitalize this sadly neglected subject. The first eleven chapters are devoted to Euclidean geometry and coordinates. In the next five, attention is given to ordered, affine, projective, absolute and hyperbolic geometry. In five more chapters, differential geometry and tensors are developed, leading to a discussion of geodesics and the topology of surfaces. A final chapter is on four dimensional geometry. The golden thread which runs through this variety of topics is an examination of the group of transformations under which the propositions of a particular geometry remain valid. This follows the program suggested by Klein (Erlangen program). The text shows an extreme care in the arrangement of propositions and their proofs, so that a great economy of space is obtained in which each subject seems to arise in the most natural manner. The treatment of Sylvester's problem of collinear points, to mention one example, has all the polish of a fine jewel. The tensor notation, to mention another, is introduced with brevity but with all the force of great clarity. The material includes a discussion of applications to crystallography, kinematics, botany and other subjects.

By a proper selection of chapters, this book can be used for a short survey

course on geometry, a variety of emphasis being possible with different selections. On the other hand, it can be made the basis of a longer course. The bibliography is not only thorough but at every stage of the book references to it for a further or more complete treatment are introduced. Indeed, these references are part of the text in every sense. The problems are appropriate and also form an integral part of the text.

Many sections are headed with quotations from a variety of sources. Where they occur, these quotations are always apposite and sometimes provocative. The reviewer enjoyed reading this book; it is a scholarly work in which wit and craftsmanship (and tender loving care) shine from every page. Its just desert should be a long, long run of popularity.

G. M. PETERSEN
University College of Swansea

Topologische Lineare Räume I. By Gottfried Köthe. Springer-Verlag, Berlin, 1960. xii+456 pp. DM 73.50; Cloth DM 78.00.

This book, which appears as vol. 107 of the well-known German "yellow series," *Die Grundlehren der Mathematischen Wissenschaften*, is certainly a worthy addition to the few but increasing number of books on linear topological spaces. The author, who over a long period of time has contributed a number of articles to this field, has set as his purpose to give a systematic presentation of the fundamental ideas, methods, and results of the theory of linear topological spaces. To this end he seems to have succeeded very well.

This very comprehensive book is in a sense self-contained in that the first chapter presents general topology in sufficient detail for the later works. Similarly, in the second chapter linear spaces for finite or infinite dimension and over arbitrary fields is given. After only about one-fourth of the book is used up with this introduction, the author then begins to deal seriously with the subject at hand. In chapter three, where the setting is real and complex linear topological spaces, the classical results of Banach are given. For the last half of the book, which is made up of three chapters, the setting is specialized to locally convex spaces. This portion of the book is very extensive. There are paragraphs discussing duality, comparisons of topologies, reflexivity, convex sets, extreme points, as well as other topics. Finally (F)-spaces, tonnelated and bornological spaces are discussed in some detail.

Admittedly the book has been strongly influenced by the French writings, especially by the two volumes of Bourbaki on the same subject. There are no exercises but there are examples that help the reader understand the theory. The book has a large index to facilitate its use as a reference and there are some ten pages of bibliography that orientates the reader in related literature.

The author indicates that in the second volume of this book he plans to treat the theory of linear mappings and to discuss important spaces and classes of spaces for the analysis. Although Hilbert spaces are included in the category of spaces given by the title there is no treatment of them in this volume and

neither is any such treatment planned in the second volume. However, the omission is intentional in view of the writings on Hilbert spaces that are already available.

E. K. McLachlan
Oklahoma State University

Introduction to Probability and Statistics. By Henry L. Alder and Edward B. Roessler. Freeman, San Francisco, 1960. vii+252 pp. $3.50.

The attempt here is to present material suitable for a one-semester course for college freshmen and sophomores (and "no valid reason why it could not be taken in high school") with two years' high school algebra as prerequisite. Problems are drawn from many specialties in which statistical methods are applied, with the recommendation that students of all these specialties take the same introductory course.

The chapters include: Organization of Data (16 pages); Summation Notation (6 pages); Analysis of Data (24 pages); Elementary Probability, Permutations, and Combinations (26 pages); The Binomial Distribution (6 pages); The Normal Distribution (14 pages); Random Sampling—Large Sample Theory (distribution of sample statistics) (14 pages); Testing Hypotheses—Significance Levels—Confidence Limits, Large Sample Methods (13 pages); Student's t-Distribution, Small Sample Methods (18 pages); The Sign Test (5 pages); Regression and Correlation (28) pages); Chi-Square Distribution (16 pages); Index Numbers (7 pages); Time Series (23 pages).

I expect the student's view of statistics after such a one-semester course would be highly mechanistic, with emphasis on finding areas under curves by using tables.

M. Maxfield
Gainesville, Florida

Essentials of Mathematics. By Russell V. Person. Wiley, New York, 1961. x+646 pp. $7.00.

This text appears to be written for persons who have had little or no secondary school mathematics, and is used by the author as a preparatory mathematics course at The Capitol Radio Engineering Institute, Washington. The five parts of the book and the number of pages devoted to each part are as follows: Arithmetic (85), Geometry (90), Algebra (256), Logarithms (56), and Trigonometry (117). The approach is the usual one of thirty years ago.

The arithmetic section deals with the fundamental operations on whole numbers and fractions, and concludes by introducing the metric system and the arithmetic process for finding square roots. The geometry presented is junior high school level informal geometry of plane and solid figures; no proofs are given. The algebra portion is much more comprehensive, and is essentially a traditional ninth-grade algebra course, with a few topics included from inter-

mediate algebra. In the logarithm section the author uses, without proof, the fundamental properties of logarithms for computation, but in the trigonometry section he takes time to prove the formulas such as that for $\sin(A+B)$. Exercises are numerous and well chosen; no answers are given.

There are enough erroneous and disturbing statements scattered throughout the book to make this reviewer question its place in *any* curriculum. As evidence, the following three quotations are given. On page 12 the author asserts that "(2)(5)(7) is the same as (2)(7)(5) or (5)(7)(2). . . . This principle is called the *associative law of multiplication.*" (Nowhere is the commutative law mentioned.) On page 130 he states that "As a secant line moves closer and closer to the side of the circle, it comes nearer and nearer to becoming a tangent line." With regard to the equation, $x+y=11$, he misinforms us on page 294 that "The values of x and y cannot be definitely determined numerically. Such an equation is called an *indeterminate* equation."

VIOLET HACHMEISTER LARNEY
State University of New York

Representation Theory of the Symmetric Group. By G. de B. Robinson. University of Toronto Press, 1961. 204 pp. $6.00.

After presenting clearly the fundamentals of the theory of characters and representations, both ordinary and modular, of an arbitrary finite group, the author turns his attention specifically to the symmetric group $\mathfrak{S}_n$. Here the Young diagram plays the fundamental role and leads explicitly to the idempotents of the ordinary representations, and to a construction of the matrices for the generating transpositions. Young's raising operator R_{it}, lattice permutations, and the hook graph are developed as basic tools in the ordinary representation theory of $\mathfrak{S}_n$, and this theory in turn is intimately related to the representation theory of the full linear group.

The characters of certain classes of $\mathfrak{S}_n$ are computed in Chapter IV, using the ideas of q-content, hook structure, q-quotient and q-core of right and skew diagrams. The modular theory is then developed in Chapters V and VI, associating a p-block with each p-core, and using r-inducing and r-restricting to sharpen the detailed study of modular representations.

Unpublished results of Robinson and Taulbee of the indecomposables and their "admitted permutations," are detailed in Chapter VII and illustrated by tables in the appendix. Results of Robinson and Diane Johnson on integral matrices for modular representations highlight the final chapter.

Throughout this welcome book a wealth of illustrative examples clarify the intricate, fascinating, and still somewhat elusive theory of the modular representations. A final bibliography of over 200 references helps to bring the interested reader up to date.

J. S. FRAME
Michigan State University

Introduction to Symbolic Logic. By A. H. Basson and C. J. O'Connor. Free Press, Glencoe, Ill., 1960. viii+175 pp. $3.00.

After a philosophical and historical introduction, pages 16–98 are devoted to propositional calculus. Pages 99–125 treat lower predicate calculus with discussion limited to monadic formulas without nesting of quantifiers. Pages 126–142 briefly consider full lower predicate calculus. A twenty-page appendix discusses classical syllogism and algebra of classes. Eight pages of exercises, mostly of the Lewis Carroll testing-valid-inferences-in-English variety, conclude the book.

The book is written with sensitivity and understanding well above average for its kind. For a student of classical philosophy and philosophic logic, it would be a good elementary introduction to some ideas of modern symbolic logic. For a student with good mathematical background or a student who wishes eventually to go beyond the most elementary ideas, the book does not go far enough or fast enough. It is better than several recent texts, however, in that it gives an accurate and proper emphasis to the topics it does include, and in that the authors make a point of qualitatively mentioning a few of the more advanced ideas that they do not include (*e.g.* decision procedures and Turing machines, infinite domains of differing cardinality). A good elementary undergraduate mathematics text in modern logic must move quickly and efficiently to a general syntactical formulation of quantifier logic. This book moves too slowly, does not in the end achieve a full formulation, and is aimed at students with traditional philosophical training rather than at mathematics students.

HARTLEY ROGERS, JR.
Massachusetts Institute of Technology

Calculus and Analytic Geometry. By Robert C. Fisher and Allen D. Ziebur. Prentice-Hall, Englewood Cliffs, N. J., 1961. xv+766 pp. $9.50.

This book is similar to the currently popular texts, but it has a few features which serve to distinguish it from the others. For example, it has more material on vectors and matrices than is normal and limits are treated quite differently than usual.

The integral is introduced with extensive discussion, which helps to clarify the concept and single it out as more important than a simple manipulative reversal of differentiation. However, I find it disturbing to read "$\int f(x)dx = F(x)$ whenever $D_x F(x) = f(x)$." With such an assertion, we would have to admit that all constants are equal. Later in the book, differential equations are briefly introduced, and at that time a comment is made regarding the constant of integration. It would have been better to insert C in each integration formula and preserve the integrity of the equality sign.

Casual use of the equals sign is exhibited again in the discussion of differentials. $D_x u dx = du$ is considered only a convenient device to avoid errors in substitution under an integral sign and "we should not treat this as a true equality

between numbers." Certainly differentials are useful and simple enough to justify inserting their definition in place of the statement that "*dx*, *dy*, *du* alone don't have meaning for us."

Vectors and three-dimensional analytic geometry appear in about the middle of the book. This is followed by a chapter on matrices which includes inverses and characteristic values of matrices. Orthogonal matrices are used to study conic sections. The properties are presented only for 3×3 matrices with an indication that some of them generalize to $n\times n$ matrices.

Instead of the standard "ϵ, δ-definitions" for limits, $\lim_{x\to a} f(x)$ is defined as the unique number L, if it exists, which is contained in every interval $J(I)$ where $I=\{x\mid 0<|x-a|<\delta\}$ and $J(I)$ is the smallest closed interval containing $\{f(x)\mid x\in I\}$. This definition has a geometric appeal and it may enable students to visualize the meaning of a limit easier than the classical approach. If nothing else, this change at least creates some variety in the course and the text is a reasonable substitute for the familiar calculus books.

James H. McKay
Michigan State University Oakland

NEWS AND NOTICES

Edited by Lloyd J. Montzingo, Jr., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor E. G. Begle, Stanford University, represented the Association at the National Conference on Curriculum Experimentation held at the University of Minnesota on September 25–28, 1961.

Dr. C. H. Wheeler, III, University of Richmond, represented the Association at the Inauguration of Dr. D. Y. Paschall as President of the College of William and Mary on October 13, 1961.

Allegheny College: Professor Emeritus M. S. Knebelman, Washington State University, has been appointed Professor; Miss Frances S. Chleboski has been appointed Instructor.

Assumption University: Dr. T. M. Klemola, University of Oulu, Finland, has been appointed Assistant Professor; Assistant Professor Elias Fakon has been promoted to Associate Professor.

Brandeis University: Professor Teruhisa Matsusaka, Northwestern University, has been appointed Professor; Assistant Professor E. M. Stein, University of Chicago, has been appointed Visiting Associate Professor; Drs. Erhard Luft, University of Bonn, Germany, and Shuichi Takahashi, Harvard University, have been appointed Research Associates; Dr. Heisuke Hironaka has been promoted to Assistant Professor; Assistant Professor E. H. Brown, Jr. has been promoted to Associate Professor; Associate Profes-

sor Oscar Goldman has been promoted to Professor; Associate Professor Maurice Auslander has been granted an Alfred P. Sloan Research Fellowship for two years.

Brown University: Assistant Professor Bruno Harris, Northwestern University, has been appointed Associate Professor; Dr. A. H. Clark, Princeton University, and Tatsuji Kambayashi, Northwestern University, have been appointed Instructors; Associate Professors David Gale, F. M. Stewart, and John Wermer have been promoted to Professors.

Butler University: Dr. Kaj Nielsen, General Motors Corporation, Indianapolis, Indiana, has been appointed Professor; Mr. Justin Wickens, Purdue University, has been appointed Instructor.

Case Institute of Technology: Associate Professor L. R. Bragg, University of West Virginia, has been appointed Assistant Professor; Assistant Professor R. M. Haber, University of Illinois, and Dr. Zakkulu Govindarajulu, University of Minnesota, have been appointed Assistant Professors.

Central State College: Dr. Laverne Loman, University of Oklahoma, has been appointed Assistant Professor; Mr. Francis Olbert, University of Mississippi, has been appointed Instructor; Miss Jane Pinkerton has returned after a one year leave of absence at Harvard University and has been promoted to Assistant Professor; Mrs. Dorothea Meagher has been named Teacher of the Year.

Colgate University: Mrs. Gertrude Pownall, Hofstra College, has been appointed Instructor; Dr. M. W. Pownall has been promoted to Assistant Professor.

Dartmouth College: Dr. J. W. Lamperti, Stanford University, has been appointed Visiting Assistant Professor; Drs. Eugene Albert, University of Virginia, and D. R. Ostberg, Rutgers University, have been appointed Research Instructors; Assistant Professor R. Z. Norman has been promoted to Associate Professor.

Denison University: Associate Professor R. A. Roberts, Ohio Wesleyan University, has been appointed Associate Professor; Mr. Donald Tichenor, Ohio University, has been appointed Instructor; Associate Professor Arnold Grudin has been granted a National Science Foundation Postdoctoral Faculty Fellowship and is spending the academic year at the Massachusetts Institute of Technology.

Eastern Illinois University: Mr. Roy Meyerholtz, University of Illinois, has been appointed Instructor; Associate Professor Lester Van Deventer has been promoted to Professor; Professor L. A. Ringenberg has been appointed Dean of the College of Letters and Sciences.

Fenn College: Mr. R. M. Stark has been appointed Assistant Professor; Miss Shirley A. Lilge, State University of Iowa, and Mr. Edgar Young, University of Cincinnati, have been appointed Instructors; Miss Yi Chang has been promoted to Assistant Professor and granted a one year leave of absence to join a computer group at Massachusetts Institute of Technology.

Fresno State College: Associate Professor A. E. Labarre, Jr., University of Idaho, has been appointed Professor and Chairman of the Department of Mathematics; Associate Professor W. A. Rees, Rice Institute, and Assistant Professor Gus Di Antonio, Duquesne University, have been appointed Assistant Professors; Major L. D. Walker, United States Air Force, and Mr. C. O. Worm, University of Idaho, have been appointed Instructors; Assistant Professor V. E. Howes is on leave of absence and will be at the University of Washington.

Georgetown University: Dr. R. L. McCoart, University of North Carolina, has been appointed Assistant Professor; Miss Brenda C. McKeon, Johns Hopkins Applied Physics Laboratories, has been appointed Instructor; Professor R. E. Ingram, University of Iowa, has been appointed Visiting Professor; Assistant Professors A. K. Aziz, J. E. Houle, and Anne Scheerer have been promoted to Associate Professors.

Illinois Institute of Technology: Associate Professor W. F. Darsow, DePaul University, has been appointed Associate Professor; Messrs Seymour Kass and Arthur

Pfeiffer have been appointed Instructors; Professor L. M. Graves, University of Chicago, has been appointed Visiting Professor; Assistant Professor J. J. Mahlberg has been promoted to Associate Professor; Dr. N. C. Petridis has been promoted to Assistant Professor.

Kansas State University: Dr. N. E. Foland, University of Missouri, has been appointed Assistant Professor; Messrs A. E. Goplen, North Dakota School of Forestry, and A. A. Richert, Nebraska Wesleyan University, have been appointed Instructors; Professor Tibor Rado, Ohio State University, has been appointed Visiting Professor.

Louisiana State University, New Orleans: Dr. W. L. Allen, Librascope, Glendale, California, has been appointed Assistant Professor; Messrs J. T. Matti, University of Detroit, A. J. Hulin, University of Southwestern Louisiana, P. S. Schnare, University of New Hampshire, Mrs. Marjorie H. Bean, Spring Hill College, and Miss Elizabeth A. Magarian, Florida State University, have been appointed Instructors.

Mississippi Southern College: Dr. P. G. Webster, Auburn University, has been appointed Associate Professor; Messrs A. L. Hare, Louisiana Polytechnic Institute, and Danny Carter, Mississippi Southern College, have been appointed Instructors; Associate Professor W. M. Sanders has been granted nine months leave to the University of Illinois; Assistant Professor Gasteon Smith has been granted nine months leave to the University of Alabama; Associate Professor Virginia Felder has been promoted to Professor.

Montana State College: Dr. Herbert Gross, University of Zurich, Switzerland, has been appointed Assistant Professor; Professor J. W. Hurst has retired as Head of the Department of Mathematics but will remain on the teaching staff as Professor; Associate Professor J. E. Whitesitt has been promoted to Professor and appointed Head of the Department of Mathematics.

Newark College of Engineering: Mr. S. G. Marx has been promoted to Instructor; Mr. W. D. Brower has been promoted to Assistant Professor.

North Texas State University: Messrs J. B. Moore, Northeast Louisiana State College, and F. R. Vest, East Texas State College, have been appointed Instructors; Assistant Professors D. F. Dawson and John Mohat have been promoted to Associate Professors.

Northwestern University: Professor Ky Fan, Wayne State University, has been appointed Professor; Associate Professor D. G. Austin, University of Miami, and Assistant Professor Jerome Sacks, Cornell University, have been appointed Associate Professors; Assistant Professor Donald LaBudde, Wayne State University, has been appointed Assistant Professor; Dr. E. D. Davis, University of Chicago, Mrs. Petee Jung, University of Florida, and Mrs. Katherine E. Shannon, Harpur College, have been appointed Instructors; Associate Professor A. H. Copeland, Purdue University, has been appointed a Visiting Associate Professor; Assistant Professor R. R. Goldberg has been promoted to Associate Professor; Dr. K. R. Mount has been promoted to Assistant Professor; Professor R. P. Boas has been awarded a President's Fellowship and will be on leave for the academic year; Associate Professor H. C. Wang has been awarded a Guggenheim Fellowship and is on leave at the Institute for Advanced Study for the academic year.

Norwich University: Messrs E. L. Marsden, University of Massachusetts, and R. L. Richardson, University of Notre Dame, have been appointed Instructors; Associate Professor E. A. Race has been appointed Acting Head of the Department of Mathematics.

Polytechnic Institute of Brooklyn: Dr. L. A. MacColl, Bell Telephone Laboratories, New York, New York, has been appointed Professor; Drs. Nathan Newman, Socony-Mobil, Clive Chester, Queens College, and Seymour Lipschutz, Glassboro State College, have been appointed Assistant Professors; Dr. Robert D'heedene, Harvard University, Messrs Abraham Smuckler, University of Pennsylvania, Porphyria Hsu, Cornell University, Martin Fried, City College of New York, and Miss Barbara Moffat, Bryn Mawr

College, have been appointed Instructors; Mr. C. W. Marshall has been promoted to Assistant Professor; Assistant Professor Pinchas Mendelson has been promoted to Associate Professor; Associate Professor Harry Hochstadt has been promoted to Professor; Professor C. M. Hebbert retired August 31, 1961; Professor R. M. Foster has retired as Chairman of the Department of Mathematics but will continue as Professor and is on leave for the academic year.

Pomona College: Assistant Professor P. B. Yale, Oberlin College, has been appointed Assistant Professor; Associate Professor Elmer Tolsted has been promoted to Professor; Professor K. L. Cooke has been appointed Chairman of the Department of Mathematics; Professor C. G. Jaeger retired as Chairman of the Department of Mathematics with the title of Professor Emeritus and has been appointed Professor at Claremont Men's College.

Quincy College: Mrs. Mary A. Snowden, University of Illinois, has been appointed Instructor; Dr. J. J. Windolph has been promoted to Assistant Professor and appointed Head of the Department of Mathematics; Professor D. G. Velesz has resigned from the teaching staff to devote full-time to his position as Registrar.

Rosary College: Sister M. Raimonda Allard has been granted a leave of absence to study at the Catholic University of America; Sister M. Mariola Dobbin has retired with the title of Professor Emeritus.

Sacramento State College: Dr. Marguerite E. Dunton, University of Colorado, has been appointed Assistant Professor; Dr. R. L. Alves has been promoted to Assistant Professor; Associate Professor S. P. Hughart is on leave for the academic year as Visiting Associate Professor at the University of California, Berkeley.

Smith College: Assistant Professor S. M. Robinson, University of Rhode Island, has been appointed Assistant Professor; Professor N. H. McCoy has been appointed to the Gates Professorship of Mathematics.

South Dakota School of Mines and Technology: Messrs M. S. Briggs, Virginia City, Montana, and Wayne Walther, San Fernando Valley State College, have been appointed Instructors.

State University of Iowa: Dr. George Burke, University of Missouri, has been appointed Instructor; Dr. Motoyoshi Sakuma has been appointed Research Associate; Drs. J. C. Hickman and J. F. Jakobsen have been promoted to Assistant Professors; Assistant Professor Steve Armentrout has been promoted to Associate Professor.

State University of South Dakota: Assistant Professor P. T. Rygg, University of Montana, has been appointed Associate Professor; Mr. Alexander Mehaffey, Jr., has been promoted to Assistant Professor.

University of Arizona: Assistant Professor J. A. Dyer, University of Texas, has been appointed Assistant Professor; Drs. M. S. Cheeman, University of California, and N. C. Giri, Stanford University, have been appointed Visiting Assistant Professors; Dr. L. M. Milne-Thomson, Mathematics Research Center, U. S. Army, has been appointed Visiting Professor.

University of California, Davis: Dr. P. W. M. John, California Research Corporation, Richmond, California, has been appointed Associate Professor; Associate Professor C. M. Fulton has been promoted to Professor.

University of California, Riverside: Professor H. D. Brunk, University of Missouri, has been appointed Professor; Messrs M. T. Boswell, Los Angeles State College, and E. S. Thomas, University of Washington, have been appointed Associates in Mathematics; Assistant Professors C. J. A. Halberg, Jr. and V. A. Kramer have been promoted to Associate Professors.

University of Chattanooga: Commander H. V. Sellers, Jr., U.S.N. (ret.), Purdue University, and Mr. C. A. Brown, Florida State University, have been appointed Assistant Professors.

University of Colorado: Dr. D. F. Rearick, University of British Columbia, has been appointed Assistant Professor; Mr. N. McC. Speake, University of Michigan, has been appointed Instructor; Dr. Ruth R. Struik, University of British Columbia, has been appointed Acting Assistant Professor; Assistant Professor Irwin Fischer is on leave for the academic year as Research Fellow at Harvard University; Associate Professor W. E. Briggs is on leave for the academic year under a National Science Foundation Fellowship at the University College, London, England; Professor Arne Magnus is on exchange for the academic year at the Institute of Technology, Trondheim, Norway, and is replaced by Professor J. O. Stubban of that Institute; Professor R. T. Lyche, Institute for Matematiske Fag, Universitetet, Blindern, Oslo, Norway, has been appointed a Visiting Professor for the academic year.

University of Dayton: Mr. S. J. Back, Research Institute, University of Dayton, Dr. J. L. Nanda, Indiana University, and Assistant Professor G. F. Speck, Suny College of Education, have been appointed Assistant Professors; Mr. P. F. Dierker, Brown University, has been appointed Instructor; Assistant Professors C. L. Keller and M. H. M. Esser have been promoted to Associate Professors.

University of Detroit: Mrs. Marie Shou-Jen Hu and Mr. C. A. Schulz, Jr., Wayne State University, have been appointed Instructors; Miss Nora Pernavs has been promoted to Assistant Professor; Dr. P. J. Reddy has been promoted to Assistant Professor and appointed Assistant Dean of the College of Arts and Sciences; Assistant Professor J. A. Mansour has been appointed Director of Registration.

University of Idaho: Assistant Professor Delmar Boyer, Fresno State College, and Dr. S. S. Mitra, University of Washington, have been appointed Assistant Professors; Assistant Professor S. A. Husain, Seattle University, has been appointed Visiting Assistant Professor; Miss Monika Aumann has been appointed Visiting Instructor; Associate Professor Hans Sagan has been promoted to Professor and appointed Head of the Department of Mathematics.

University of Illinois: Associate Professor Mary E. Hamstrom, Goucher College, and Assistant Professor J. H. Walter, University of Chicago, have been appointed Associate Professors; Drs. K. I. Appel, Institute of Defense Analyses, and P. W. Mikulski, University of California, Berkeley, have been appointed Assistant Professors; Drs. Zbegniew Ciesielski, Cornell University, and Fumiyuki Maeda, Yale University, have been appointed Research Associates; Drs. F. M. Djorup, Cornell University, D. J. Eustice, National Aeronautics and Space Administration, Cleveland, Ohio, J. J. Harvey, Tulane University, Yuji Ito, Yale University, L. R. McCulloh, Wright-Patterson AFB, A. L. Peressini, Washington State University, R. R. Rao, Calcutta, India, Julius Smith, University of California, Berkeley, J. E. Wetzel, Stanford University, and Patricia A. Tucker, University of Wisconsin, have been appointed Instructors; Visiting Professor Noboru Ito, University of Chicago, has been appointed Visiting Professor; Assistant Professor Lily Seshu, Harpur College, has been appointed Visiting Assistant Professor; Dr. J. J. Rotman has been promoted to Assistant Professor; Assistant Professor Josephine M. Chanler has been promoted to Associate Professor.

University of Michigan: Professor P. R. Halmos, University of Chicago, has been appointed Professor; Dr. D. J. Lewis, University of Cambridge, England, has been appointed Associate Professor; Dr. C. Pommerenke, University of Gottingen, Germany, has been appointed Assistant Professor; Drs. R. C. O'Neill, Purdue University, J. H. Gillilan, University of Illinois, and J. L. Goldberg, Bell Telephone Laboratories, Murray Hill, New Jersey, have been appointed Instructors; Drs. H. W. Knobloch, University of Munich, Germany, and J. R. Boen, University of Chicago, have been appointed Lecturers; Professor G. af Hallstrom, Abo Akademi, Finland, has been appointed Visiting Professor; Associate Professor K. W. Gruenberg, Queen Mary College, England, has been appointed Visiting Associate Professor; Associate Professor J. G.

Wendel has been promoted to Professor; Assistant Professors A. B. Clarke, D. R. Hughes and J. L. Ullman have been promoted to Associate Professors; Drs. N. J. Hicks, C. N. Lee, M. S. Ramanujan and R. H. Rosen have been promoted to Assistant Professors; Professor H. C. Carver retired in September 1961.

University of Nebraska: Dr. Albert Zechmann, Iowa State University, has been appointed Assistant Professor; Associate Professor Edwin Halfar has been promoted to Professor; Assistant Professors Bernard Harris, W. E. Mientka and H. H. Schneider have been promoted to Associate Professors; Professor M. A. Basoco has been awarded a $1,000 prize for Distinguished Teaching by the University of Nebraska Foundation.

University of Ottawa: Drs. Taqdir Husain, Syracuse University, and Heinrich Kleisli, University of Montreal, have been appointed Assistant Professors.

University of Puerto Rico, Rio Piedras: Dr. Gloria Diaz-Gomez and Mr. Warren Novak have been appointed instructors; Drs. Hilda Madel, Ernesto Smith, and Margarita Rodriguez have been appointed Lecturers; Assistant Professor J. R. Padro has been promoted to Associate Professor; Dr. Hilda M. Villemil has been promoted to Assistant Professor; Professor Francisco Garriga will represent Puerto Rico at the Inter-American Conference on Mathematical Education in Bogota, Colombia in December 1961.

Vanderbilt University: Assistant Professor Diran Sarafyan, Utah State University, has been appointed Associate Professor; Dr. D. B. Coleman, Michigan State University, has been appointed Assistant Instructor.

Washington University: Associate Professor Guido Weiss, DePaul University, has been appointed Associate Professor; Assistant Professor C. D. Gorman, Institute of Mathematical Sciences, has been appointed Assistant Professor; Assistant Professor Mary B. Weiss, DePaul University, has been appointed Lecturer; Assistant Professor R. N. Kesarwani, Panjab University, India, has been appointed Visiting Assistant Professor; Dr. J. C. Merlo, University of Buenos Aires, Argentina, has been appointed Visiting Research Associate; Professor I. I. Hirschman, Jr. has been appointed Chairman of the Department of Mathematics; Associate Professor Allen Devinatz has been promoted to Professor; Assistant Professors H. M. Feldman and A. E. Nussbaum have been promoted to Associate Professors.

Western Washington State College: Mr. M. G. Billings, University of Montana, has been appointed Instructor; Assistant Professor S. T. Rio has been promoted to Associate Professor.

Wisconsin State College, River Falls: Messrs J. L. Brown, Rapid City, South Dakota, and F. C. Stewart, University of Minnesota, have been appointed Instructors.

Worcester Polytechnic Institute: Associate Professor J. P. Van Alstyne, Hamilton College, has been appointed Associate Professor; Messrs P. E. Shakir, Raytheon Manufacturing Company, and J. J. Kaul, Sylvania Electric Products Company, have been appointed Instructors; Drs. V. C. Connolly, R. C. Scott, G. C. Branche and Bernard Howard have been promoted to Assistant Professors.

Dr. F. D. Alexander, Stephen F. Austin State College, has been promoted to Assistant Professor.

Mr. P. H. Anderson, Purdue University, has been appointed Assistant Professor at Montclair State College.

Assistant Professor F. G. Asenjo, Georgetown University, has been appointed Associate Professor at the University of Southern Illinois.

Mr. R. H. Balomenos, Harvard University, has been appointed Assistant Professor at the University of New Hampshire.

Associate Professor R. G. Bartle, University of Illinois, is on leave for the first semester at the University of California, Berkeley.

Professor P. T. Bateman, University of Illinois, is on leave at the University of Pennsylvania.

Dr. W. H. Beyer, Virginia Polytechnic Institute, has been appointed Assistant Professor at the University of Akron.

Assistant Professor F. T. Birtel is on leave from Ohio State University and has been appointed Research Associate and Lecturer at Yale University for the academic year.

Dr. Alfred Blumstein, Cornell Aeronautical Laboratory, Buffalo, New York, has accepted a position with the Institute for Defense Analyses, Washington, D.C.

Dr. S. E. Bohn, University of Nebraska, has been appointed Assistant Professor at Bowling Green State University.

Professor David Bourgin, University of Illinois, will be on leave for the second semester at the Institute of Mathematics, Rome, Italy.

Assistant Professor J. C. Bradford, Abilene Christian College, has accepted a position in the Advanced Analysis Department of United ElectroDynamics, Pasadena, California.

Associate Professor D. L. Burkholder, University of Illinois, is on leave for the first semester at the University of California, Berkeley.

Professor K. A. Bush, University of Idaho, has been appointed Professor at Washington State University.

Assistant Professor J. B. Butler, Jr., University of Arizona, has been appointed Associate Professor at Portland State College.

Assistant Professor Hsim Chu, Northwestern University, has been appointed Assistant Professor at the University of British Columbia.

Lt. Col. R. E. Clark, Virginia Military Institute, has accepted a position as Mathematician with Babcock and Wilcox Company, Lynchburg, Virginia.

Assistant Professor George Craft, Denison University, has been appointed Assistant Professor at Harpur College.

Mr. R. T. Douglass, University of Michigan, has been appointed Instructor at Kent State University.

Associate Professor Meyer Dwass, Northwestern University, has been appointed Professor at the University of Minnesota.

Mr. E. J. Eckert, Los Angeles State College, has been promoted to Assistant Professor.

Associate Professor Joanne Elliott, Barnard College, has received a National Science Foundation Senior Postdoctoral Fellowship and will be on leave at the Institute for Advanced Study and in Paris, France.

Professor M. P. Emerson, Southwest Missouri State College, has been appointed Chairman of the Department of Mathematics at Kansas State Teachers College.

Assistant Professor W. C. Fox, Northwestern University, has been appointed Associate Professor at the State University of New York, Long Island Center.

Mr. R. R. Gero, University of Massachusetts, has accepted a position as Assistant Engineer with the Sperry Gyroscope Company, Great Neck, Long Island, New York.

Assistant Professor C. B. Germain, University of Manitoba, has been appointed Assistant Professor at the College of St. Thomas.

Associate Professor Sudhish Ghurye, Northwestern University, has been appointed Associate Professor at the University of Minnesota.

Dr. J. B. Goebel, Oregon State University, has accepted a position as Mathematician in the Hanford Laboratories of the General Electric Company, Richland, Washington.

Dr. A. J. Goldman has been appointed Chief of the Operations Research Section of the Applied Mathematics Division at the National Bureau of Standards, Washington, D. C.

Mr. Neil Grabois, University of Pennsylvania, has been appointed Instructor at

Lafayette College.

Mr. C. A. Green, University of Wisconsin, has been appointed Instructor at Ohio University.

Dr. A. L. Gropen, Wellesley College, has been promoted to Assistant Professor.

Dr. A. E. P. Haron, Sorbonne, Paris, has been appointed to the faculty of Sarah Lawrence College.

Mr. J. E. Homer, Jr., Remington Rand Univac, St. Paul, Minnesota, has been appointed Instructor at St. John's University, Collegeville, Minnesota.

Miss Nancy Hsieh, Lehigh University, has been appointed Instructor at the Western College for Women.

Mr. R. A. Jacobson, South Dakota School of Mines and Technology, has been appointed Instructor at South Dakota State College.

Assistant Professor W. D. James, Fresno State College, has accepted a position with the Philco Corporation, Palo Alto, California.

Assistant Professor John Jewett, University of Georgia, has been promoted to Associate Professor.

Mr. D. E. Johnson, Argonne National Laboratories, Argonne, Illinois, has been appointed Instructor at North Central College.

Professor Mark Kac, Cornell University, has been appointed Professor at the Rockefeller Institute, New York City, New York.

Mr. M. D. Landau, Philadelphia Textile Institute, Philadelphia, Pennsylvania, has been appointed Assistant Professor at Glassboro State College.

Mr. B. R. Lane, University of Chattanooga, has been appointed Instructor at Vanderbilt University.

Assistant Professor L. H. Lange, San Jose State College, has been promoted to Associate Professor.

Assistant Professor B. L. McAllister is on sabbatical leave from the South Dakota School of Mines and Technology and is studying at the University of Wisconsin.

Dr. G. S. McCarty, Jr., University of California, Los Angeles, has been appointed Instructor at the University of Chicago.

Mr. R. A. Meili, Central Florida Junior College, has been appointed Instructor at St. Lawrence University.

Mr. H. G. Moore, Purdue University, has been appointed Assistant Professor at Brigham Young University.

Assistant Professor J. D. Neff, Case Institute of Technology, has been appointed Assistant Professor at Georgia Institute of Technology.

Mr. Samuel Pasiencier, Northwestern University, has been appointed Assistant Professor at Lake Forest College.

Associate Professor L. C. Peck, Ohio Wesleyan University, has been appointed Associate Professor at Miami University.

Colonel K. S. Purdie, Virginia Military Institute, retired September 1, 1961.

Mr. T. J. Robertson, University of Missouri, has been appointed Assistant Professor at Cornell College.

Associate Professor Alex Rosenberg, Northwestern University, has been appointed Professor at Cornell University.

Associate Professor W. T. Scott, Northwestern University, has been appointed Professor at Arizona State University.

Sister Madeleine Sophie, University of Illinois, has been appointed Instructor at Alverno College.

Professor D. E. South, University of Florida, has been appointed Visiting Professor at Florida Presbyterian College.

Mr. J. L. Spurgeon, South Dakota School of Mines and Technology, has accepted a

position as Associate Engineer with the Boeing Company, Seattle, Washington.

Mr. D. H. Staley, Oberlin College, has been appointed Assistant Professor at Ohio Wesleyan University.

Associate Professor O. E. Stanaitis, St. Olaf College, has been granted a sabbatical leave for study at Stanford University for the academic year 1961–62.

Assistant Professor Konrad Supronowicz, University of Idaho, has been appointed Associate Professor at Utah State University.

Mr. R. J. Thomas, DePauw University, has been promoted to Assistant Professor and will be on leave for the academic year 1961–62 under a Danforth Fellowship.

Mr. John Van Iwaarden, Michigan State University, has been appointed Instructor at Hope College.

Mr. G. B. Williams, University of Nebraska, has been appointed Instructor at the University of the Pacific.

Professor Emeritus J. E. Burnam, Hardin-Simmons University, died June 27, 1961. He was a member of the Association for forty years.

Professor C. C. MacDuffee, University of Wisconsin, died on August 21, 1961. He was a member of the Association for forty years. Professor MacDuffee served the Association as President (1945–46) and as author of the Carus Monograph entitled *Vectors and Matrices*.

Professor Glenn James, Arroyo Grande, California, died on September 1, 1961. He was a member of the Association for forty-four years.

INTERNATIONAL CONGRESS OF MATHEMATICIANS 1962

At the invitation of the Swedish National Committee for Mathematics and the Swedish Mathematical Society, the International Congress of Mathematicians will meet in Stockholm from August 15th to August 22nd, 1962.

The Executive Committee is inviting a number of mathematicians to deliver one-hour and half-hour addresses; the former will be in the nature of general surveys of recent developments in the different fields of mathematics and are intended also for non-specialists. There will also be daily sessions devoted to ten-minute communications.

All fields of mathematics, including Probability and Mathematical Statistics, Mathematical Physics and Numerical Analysis, will be covered. A special session for Education will be organized in collaboration with the International Commission for Mathematical Instruction.

A programme of entertainments and excursions is being planned.

There will be two categories of membership of the Congress:

Ordinary members, who will be entitled to participate in the scientific and social activities of the Congress and to receive the Proceedings of the Congress, and

Associate members accompanying ordinary members of the Congress, who will be entitled to many of the privileges of ordinary membership, but will not participate in the scientific programme and will not receive the Proceedings.

Those who wish to receive further information about the Congress are requested to communicate their name and full address to Ragnar Thorn, Secretary, International Congress of Mathematicians, Djursholm 1, Sweden.

REGISTER OF SCIENTISTS INTERESTED IN OVERSEAS ASSIGNMENTS

American participation in the educational and economic development of other countries is likely to continue on an increasing scale during the next few years. In anticipation of the growing need for personnel under international programs in which it cooperates, the National Academy of Sciences-National Research Council is compiling a register of American scientists and other specialists who are interested in the possibilities of

assignments abroad for periods ranging from several weeks to two years.

Assignments become available irregularly throughout the year, and they vary greatly with respect to location, duration, stipends, and responsibilities. Some programs sponsored by private foundations require scientists of high competence and reputation for short-term lecturing and consultative duties in a single country or in several countries. A number of government-sponsored projects call for specialists who are available for two year periods, have had previous experience in certain geographical areas, and are fluent in Spanish or French. Also, under the exchange program authorized by the Fulbright and Smith-Mundt Acts, younger as well as established science educators are welcomed as lecturers at many African, Asian, and Latin American colleges and universities.

Persons who wish to be considered for any of such assignments are asked to fill out a special form, available upon request from the *Committee on International Exchange of Persons*, 2101 Constitution Avenue, N.W., Washington 25, D. C. Completion and return of the form will not constitute an application, but it would ensure a person's consideration for openings in his field.

The Register is intended specifically for specialists in the biological and physical sciences and related technologies. However, the Committee will be happy to receive inquiries from persons in other fields.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE JUNE MEETING OF THE NORTHEASTERN SECTION

The eighth annual meeting of the Northeastern Section of the Mathematical Association of America was held at the University of Vermont, Burlington, Vermont, on June 20, 1961. Professor D. E. Christie, Chairman of the Section, presided at the morning session and Professor R. A. Rosenbaum, Vice-Chairman of the Section, presided at the afternoon session. There were 55 persons present, including 48 members of the Association.

At the annual business meeting of the Section the following officers were elected: Chairman, Professor R. A. Rosenbaum, Wesleyan University; Vice-Chairman, Professor M. E. Munroe, University of New Hampshire; Secretary-Treasurer, Mr. R. S. Pieters, Phillips Academy.

The program of the meeting was as follows:

1. *Nonlinear problems in applied mathematics*, by Professor L. N. Howard, Massachusetts Institute of Technology.

2. *Geometric algebra*, by Professor Ernst Snapper, Indiana University.

3. *Invertible spaces*, by Professor J. G. Hocking, Michigan State University.

4. *Some dissection curiosities*, by Professor Howard Eves, University of Maine.

5. *Pre-graduate training: A progress report for the CUPM*, by Professor J. C. Moore, Princeton University.

R. S. PIETERS, *Secretary*

MAA STUDIES IN MATHEMATICS

The first volume "Studies in Modern Analysis" of a new series of books entitled *MAA Studies in Mathematics* will be published in January 1962. This volume of about 175 pages will contain articles by E. J. McShane, M. H. Stone, E. R. Lorch, and Casper Goffman, with a preface by R. P. Dilworth and an introduction by R. C. Buck, the editor of the volume.

Each member of the Association may purchase one copy of each volume of the *Studies*. The price of Volume 1 is $2. Orders with remittance should be sent to the Buffalo office of the Association.

Additional copies and copies for nonmembers may be purchased from Prentice-Hall, Inc., Englewood Cliffs, N. J. A remittance of $4 should accompany each order. Bookstores may order copies in quantity under their usual terms.

PROPOSED AMENDMENTS TO THE BY-LAWS OF THE MAA

At the meeting of the Board of Governors at Stillwater, Oklahoma, on August 28, 1961, the Secretary was instructed to prepare the necessary amendments to the By-Laws to provide for members of the Finance Committee to be ex-officio members of the Board of Governors, for the President to be an ex-officio member of the Finance Committee and to be Chairman of the Executive Committee and of the Finance Committee, for the office of President-elect to be established, etc. (for details see the report on the meeting of the Board of Governors on page 952 of the November issue of the MONTHLY).

The Secretary now announces that at the business meeting of the Association to be held at the Sheraton-Gibson Hotel in Cincinnati, Ohio, on Thursday, January 25, 1962, motions will be made to amend the By-Laws as follows (these amendments to be effective at the conclusion of the Annual Meeting in January 1962, with the provision that the first President-elect shall be elected in 1963, take office in January 1964 and become President in January 1965):

A. That Article III, Section 1, be amended to read:

The Officers of the Association shall be a President, a President-elect (only during a year immediately prior to the expiration of a President's term), a Past-President (only during a year immediately following the expiration of a President's term), a First Vice-President, a Second Vice-President, an Editor-in-Chief of the Official Journal (hereinafter called the "Editor"), a Secretary, a Treasurer, and an Associate Secretary.

B. That Article III, Section 2, be amended to read:

There shall be a Board of Governors (hereinafter called "the Board"), to consist of the officers, the Ex-Presidents for terms of six years after the expiration of their respective presidential terms, the members of the Finance Committee, and of additional elected members . . . (otherwise the the same as Article III, Section 2, in the present By-Laws).

C. That Article III, Section 3, be amended to read:

There shall be an Executive Committee, advisory to the Board, and consisting of the President, the President-elect (only during a year immediately preceding the expiration of a President's term), the Past-President (only during a year immediately following the expiration of a President's term), the two Vice-Presidents, . . . (otherwise the same as Article III, Section 3, in the present By-Laws).

D. That the last sentence of Article III, Section 7, be amended to read:

This committee shall consist of five members, including the President, the Secretary, and the Treasurer.

E. That Article III, Section 8(b), be amended to read:

The membership at large shall elect biennially a President-elect for a term of one year and a First Vice-President for a term of two years, and shall elect annually two Governors for terms of three years. The President-elect shall become President for a two-year term at the expiration of his one-year term as President-elect and shall become Past-President for a one-year term at the expiration of his term as President.

F. That the first sentence of Article III, Section 8(e), be amended to read:

The President shall be ineligible for reëlection as President-elect or as President.

G. That the word "President" in the third sentence of Article III, Section 8(g), be replaced by "President-elect."

H. That Article III, Section 9, be amended by inserting between the two sentences of that section the following sentence:

He shall be Chairman of the Executive Committee and of the Finance Committee.

HENRY L. ALDER, *Secretary*

ACKNOWLEDGMENT

The Editors wish to acknowledge the services of the following persons, not members of the editorial staff, who have assisted the Editors by refereeing manuscripts during the past year.

H. L. Alder, B. H. Arnold, H. G. Ayre, Lida K. Barrett, L. C. Barrett, Richard Bellman, Gerald Berman, L. N. Bidwell, L. M. Blumenthal, R. P. Boas, W. W. Boone, T. A. Botts, A. V. Boyd, D. L. Boyer, W. G. Brady, H. W. Brinkmann, J. O. Brooks, James Brown, Leonard Carlitz, D. G. Chapman, N. A. Court, H. S. M. Coxeter, Helen F. Cullen, Frederic Cunningham, Jr., P. J. Davis, Douglas Derry, D. J. Dessart, S. P. Diliberto, N. J. Divinsky, L. E. Dubins, Jacqueline P. Evans, William Feller, D. T. Finkbeiner II, Harley Flanders, L. R. Ford, G. E. Forsythe, Tomlinson Fort, J. S. Frame, S. P. Franklin, Orrin Frink, Jr., C. M. Fulton, Karl Goldberg, Michael Goldberg, B. Good, R. M. Good, A. W. Goodman, H. W. Gould, E. E. Grace, L. M. Graves, Louis Green, Newcomb Greenleaf, Edwin Halfar, P. R. Halmos, Frank Harary, D. K. Harrison, L. A. Henkin, Edwin Hewitt, E. H. C. Hildebrandt, A. S. Householder, Dale Husemoller, B. W. Jones, Mark Kac, R. E. Kalaba, Wilfred Kaplan, J. L. Kelley, L. M. Kells, P. J. Kelly, M. S. Klamkin, J. W. Lamperti, Rose Lariviere, G. E. Latta, D. H. Lehmer, Marguerite Lehr, Norman Levine, Harry Levy, W. S. Loud, L. L. Lowenstein, E. J. McShane, C. C. MacDuffee, Morris Marden, Kenneth May, Karl Menger, B. E. Meserve, L. Mirsky, E. E. Moise, Leo Moser, D. C. Murdoch, Albert Newhouse, T. A. Newton, Albert Nijenhuis, J. C. C. Nitsche, Ivan Niven, R. Z. Norman, C. D. Olds, R. C. Osborn, R. R. Otter, Sam Perlis, Edmund Pinney, H. O. Pollak, E. D. Rainville, R. A. Restrepo, D. E. Richmond, John Riordan, R. M. Robinson, R. A. Rosenbaum, Francis Scheid, E. M. Scheuer, Pincus Schub, Hans Schwerdtfeger, R. T. Selley, W. H. Simons, Maurice Sion, J. Laurie Snell, Louis Solomon, A. H. Sprague, H. E. Stelson, Rothwell Stephens, B. M. Stewart, R. R. Stoll, W. L. Strother, Gabor Szegö, G. B. Thomas, Jr., R. M. Thrall, H. S. Thurston, M. L. Tomber, Leonard Tornheim, H. G. Tucker, W. R. Utz, F. A. Valentine, K. Venkannayah, D. D. Wall, A. D. Wallace, J. L. Walsh, Morgan Ward, W. H. Warner, W. M. Whyburn, Albert Wilansky, R. J. Wisner, J. W. Woll, Jr., R. C. Wrede, Jr., Max Wyman, R. C. Yates.

EDITORIAL

This issue marks the end of my term as editor of the MONTHLY and I cannot let the occasion pass without paying brief tribute to those who have done much to lighten my

work. First of all, I want to express my appreciation to my associate editors and to the referees. Then, I must thank Professor Carl B. Allendoerfer and Mrs. Helen Zuckerman for their guidance and advice at the beginning of my term, Professor Harry M. Gehman for his careful proofreading (he usually found at least one error that the rest of us missed), and my editorial assistant, Mrs. Catherine Easto, for her great help with the many details involved in the publication of the MONTHLY. Finally, I am greatly indebted to a number of people at George Banta Company, particularly Mr. Herbert D. Hartung and the mathematical compositors (whom I know only from their initials at the top of a galley).

Beginning with the January 1962 issue, the MONTHLY will be under the careful and expert guidance of Professor F. A. Ficken. To him and his board of associate editors I extend my very best wishes.

RALPH D. JAMES

CALENDAR OF FUTURE MEETINGS

Forty-fifth Annual Meeting, Sheraton-Gibson Hotel, Cincinnati, Ohio, January 24–26, 1962.

Forty-third Summer Meeting, University of British Columbia, Vancouver, August 27–29, 1962.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Chatham College, Pittsburgh, Pennsylvania, May 5, 1962.

ILLINOIS, North Central College, Naperville, May 11–12, 1962.

INDIANA, Butler University, Indianapolis, May 5, 1962.

IOWA, Wartburg College, Waverly, April 13–14, 1962.

KANSAS, Bethel College, North Newton, April 28, 1962.

KENTUCKY, University of Kentucky, Lexington Spring, 1962.

LOUISIANA-MISSISSIPPI, Tulane University, New Orleans, Louisiana, February 16–17, 1962.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, March 24, 1962.

MINNESOTA

MISSOURI, Missouri School of Mines, Rolla, Spring, 1962.

NEBRASKA, University of Nebraska, Lincoln, April 13–14, 1962.

NEW JERSEY, Rutgers, The State University, New Brunswick, November 3, 1962.

NORTHEASTERN, November 24, 1962.

NORTHERN CALIFORNIA, University of California, Davis, January 13, 1962.

OHIO

OKLAHOMA

PACIFIC NORTHWEST, Western Washington College, Bellingham, June 14, 1963.

PHILADELPHIA

ROCKY MOUNTAIN, South Dakota School of Mines, Rapid City, May 4–5, 1962.

SOUTHEASTERN, Woman's College, University of North Carolina, Greensboro, March 30–31, 1962.

SOUTHERN CALIFORNIA, Long Beach State College, March 10, 1962.

SOUTHWESTERN

TEXAS, Rice University, Houston, April 1962.

UPPER NEW YORK STATE, Clarkson College of Technology, Potsdam, Spring, 1962.

WISCONSIN, Marquette University, Milwaukee, May 12, 1962.

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GANDHI, J. M. Nonvanishing of Ramanujan's τ-function, 757–760.

GILBERT, E. N. Design of mixed doubles tournaments, 124–131.

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HORADAM, A. F. A generalized Fibonacci sequence, 455–459.

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HORADAM, E. M. Arithmetical functions of generalized primes, 626–629.

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SLATER, MICHAEL. A single postulate for groups, 346–347.

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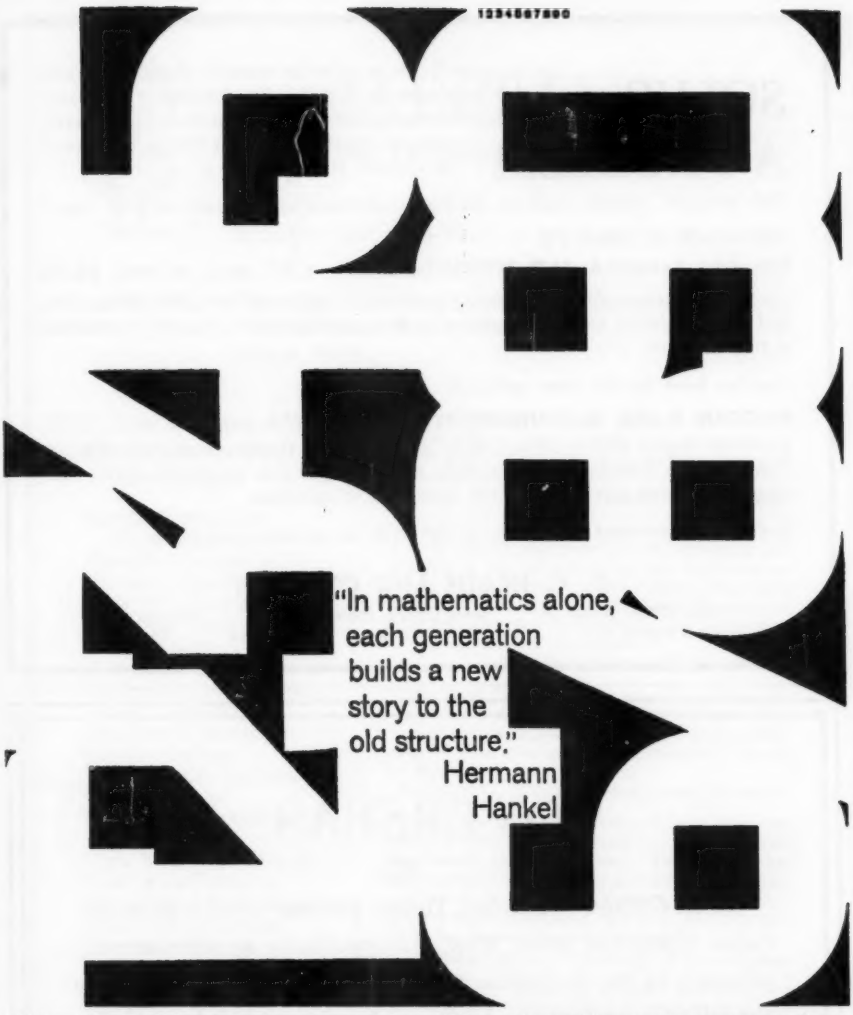

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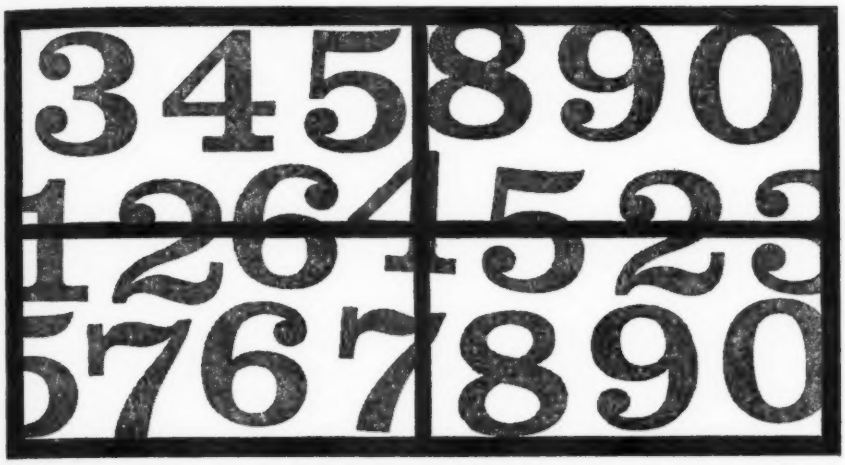

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